CHAMBERS'S EDUCATIONAL COURSE

THE STANDARD ALGEBRA

BY

GEORGE WYLLIE, M.A.

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PREFACE.

In the explanatory part of this work, principles common to both algebra and arithmetic have in some cases been taken for granted; but the peculiarities of algebra, especially those relating to signs, formulæ, abbreviated methods, &c., have received special attention. Where it seemed both possible and desirable, the concrete form of explanation has been preferred to the abstract, and upwards of 200 *illustrative examples* have been worked out in full.

The *Exercises* number upwards of 3000—several times as many as are usually given within the same field, even in much larger text-books; and, lest even so many should be too few, none have been borrowed from other works, so that those given here may, if necessary, be supplemented from other sources by exercises similar in form, yet not the same.

With a view to careful graduation, almost all the sums have been worked out in the way a beginner is likely to work them. Exercises have been set in algebraic dictation, as in Exercises I. IV. V. VI. XXVII.; and an unusually large number of very simple exercises has been given under each rule, in order that a pupil may not, on beginning it, be disconcerted by the mere mechanical difficulty of working. Many of these, for example Exercises VII. IX. XII. XVI. XVII. XLV. XLVII.–XLIX. are meant to serve for mental exercise in class-work. Great care has been taken to secure accuracy in the answers. A note of any error that may be detected will be thankfully received by the Publishers.

The work, though specially adapted to the requirements of schools under inspection, will be found suitable for elementary work in schools generally, for pupil-teachers, and for students preparing for university local examinations, or for the first and second stages of the examinations in mathematics held by the Science and Art Department.

PREFACE TO THE NEW EDITION.

In this edition the chapters on General Results in Multiplication and Resolution into Factors have been re-written, and the exercises on them greatly increased in number; many additional examples have been given in Exercises XXII. and XXXIII.; the chapter on Simple Equations and Problems has been subdivided and largely extended; and sets of Test Exercises for revisal have been added.

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STANDARD ALGEBRA—PART I.

NOTATION.

1. ALGEBRA is the science in which we reason about known and unknown quantities, using symbols to represent the quantities, and signs to shew their relations to each other.

The symbols employed are the letters of the alphabet. Usually, but not always, the first letters a, b, c, &c., stand for known quantities; the last letters, x, y, z, for unknown quantities. The signs employed are explained below.

Arithmetic also uses signs and symbols. Its signs are the same as those of algebra; its symbols are the figures 1, 2, 3, 4, &c. These, however, represent only known quantities, and therefore with known quantities alone can arithmetic deal. Algebra differs mainly from arithmetic in using other symbols besides these, in using them to represent both known and unknown quantities, and in dealing with the unknown just as if they were known. How it does so will be seen as we go on.

An algebraical letter may represent different quantities in different sums, but always stands for the same quantity in the same sum. Thus, if a stand for 5 in a given sum, it stands for 5 all through that sum, although it may stand for 6, 7, or some other quantity in another sum. Of course different letters in the same sum always stand for different quantities.

2. The sign $=$ means that the quantities between which it stands are *equal*.

It is called the *sign of equality*, and $a = b$ is read 'a is equal to b,' or 'a equals b.'

3. The sign of addition, $+$, called *plus* (the Latin word for *more*), means that the quantity before which it stands is to be *added*.

Thus, $a + b$ is read 'a plus b,' and means b added to a, or the *sum* of a and b. Similarly, $a + b + c$ is read 'a plus b plus c,' and means b added to a, and c added to their sum. If $a = 12$, $b = 5$, and $c = 3$, $a + b + c = 20$.

4. The sign of subtraction, $-$, called *minus* (the Latin word for *less*), means that the quantity before which it stands is to be *subtracted*.

Thus, $a - b$ is read 'a minus b,' and means b subtracted from a. Similarly, $a - b - c$ is read 'a minus b minus c,' and means b subtracted from a, and c subtracted from the result. If $a = 12$, $b = 5$, and $c = 3$, $a - b - c = 4$.

5. The sign of multiplication, $\times$, means that the quantities between which it stands are to be *multiplied* together. Sometimes a point (.) is used instead of $\times$, but most frequently the quantities are merely written in *close succession* without any sign between them.

Thus, $a \times b \times c$, $a.b.c$, and abc all mean that a is to be multiplied by b, and the result multiplied by c. The first and second forms are read 'a into b into c;' the third, abc.

Where *letters only* occur, or *letters preceded by a number*, $\times$ or the point is seldom employed, close succession being quite sufficient to indicate their multiplication. We therefore do not write $a \times b \times c$, $12 \times a \times b \times c$, or $a.b.c$, $12.a.b.c$, but merely abc, $12abc$. If $a = 2$, $b = 3$, and $c = 4$, $12abc = 288$.

Where *numbers only* occur, close succession does not indicate multiplication at all, and the point is used to do so only where it cannot be mistaken for the decimal point. Thus $432 = 400 + 30 + 2$, not $4 \times 3 \times 2$. So, also, 4.3 (with the point on the line) $= 4 \times 3$, and $4.3.2 = 4 \times 3 \times 2$; but $4{\cdot}3$ (with the point above the line) $= 4$ *decimal* 3; that is, $4\frac{3}{10}$.

6. The sign of division, $\div$, means that the quantity *before* it is to be *divided by* the quantity *after* it. More frequently the quantity to be divided is placed above the other with a line between them.

Thus, $a \div b$ and $\frac{a}{b}$ both mean a divided by b, and are both read 'a by b.' So, too, $a + b \div a - b$ and $\frac{a+b}{a-b}$ both mean $a + b$ divided by $a - b$, and are read '$a + b$ by $a - b$.' If $a = 12$,

and $b = 8$, $\frac{a+b}{a-b} = \frac{12+8}{12-8} = \frac{20}{4} = 5$. It will afterwards be seen that the form $\frac{a+b}{a-b}$ is preferable to $a + b \div a - b$.

Exercise I.

1. Write (1) ab, cd, $bcde$, $2cf$, $128acd$ in other two forms; (2) 532, 768, 349, 1568, 2304, 5060 in their uncontracted form; (3) $b \div c$, $de \div ef$, $24ab \div 5cd$, $b + c \div c - d$, $5ab - 4cd \div 8de + 3bc$ in another form.

2. Read each sum in Exercises 2 and 3, using instead of signs (1) the words *add, subtract, multiplied by, divided by*; (2) the words *plus, minus, into, by*.

3. Write down each of these sums when read to you.

Exercise II.

If $a = 6$, $b = 5$, $c = 4$, $d = 3$, $e = 1$, $f = 0$, find the numerical values of the following expressions.

Example.

$$a + b - 2c + 12cd - 11de - 7cf$$
$$= 6 + 5 - (2 \times 4) + (12 \times 4 \times 3) - (11 \times 3 \times 1) - (7 \times 4 \times 0)$$
$$= 6 + 5 - 8 + 144 - 33 - 0 = 114.$$

The need for enclosing 2×4, &c., in brackets is explained in section 10.

The pupil should exercise the utmost care in putting down all his work fully, intelligibly, and neatly. He may follow the models given here and in subsequent exercises, or any others his teacher may prefer.

1. $a + b + c + d + e + f$.
2. $a + b + c - d - e - f$.
3. $a + b - c - d - e + f$.
4. $a - b + c - d + e - f$.
5. $ab + cd + ef$.
6. $ac - bd + af$.
7. $ad - ce - bf$.
8. $2ab - 2cf - 2bd$.
9. $3ce - 5df + 7be$.
10. $abc - bcd + def - bde$.
11. $ace + bdf + ade - bce$.
12. $2abc - 3cde - 4def$.
13. $abcd - 3adf + 5ace$.
14. $8acd + bce - 12bde$.
15. $abde - bcde + acdf + abcd$.
16. $adef + abce - bcde + acde$.
17. $2abcd - 3bcdf + 5abde - 7bcde$.
18. $24ad - 13af + 17bd - ab$.
19. $33df + 64de + 13cd - 28be$.
20. $128cd - 54ce + 93def - 67bd$.

Exercise III.

If $a = 8$, $b = 6$, $c = 5$, $d = 4$, $e = 3$, $f = 2$, and $g = 0$, find the numerical values of the following expressions.

Example.

$$bcd - \frac{a-b}{c-d} - \frac{2ab - 4de}{3bc - 7bf}$$

$$= (6 \times 5 \times 4) - \frac{8-6}{5-4} - \frac{(2 \times 8 \times 6) - (4 \times 4 \times 3)}{(3 \times 6 \times 5) - (7 \times 6 \times 2)}$$

$$= 120 - \frac{2}{1} - \frac{96 - 48}{90 - 84} = 120 - 2 - \frac{48}{6} = 118 - 8 = 110.$$

1. $\frac{a}{d} + \frac{b}{e} - \frac{d}{f} + \frac{a}{f}$.
2. $\frac{5a}{2c} - \frac{6c}{5e} - \frac{12c}{5b} + \frac{14d}{7a}$.
3. $\frac{bc}{ef} - \frac{ab}{de} + \frac{bd}{ae} - \frac{ag}{bf}$.
4. $\frac{3cd}{5bf} + \frac{4ab}{8df} - \frac{5ad}{2ac} - \frac{12af}{16de}$.
5. $\frac{abc}{def} - \frac{ade}{bf} + \frac{bcd}{ae} - \frac{ace}{bd}$.
6. $\frac{15bde}{9ac} + \frac{25aef}{bcd} - \frac{18adf}{32be} - \frac{27acd}{15bef}$.
7. $\frac{a+d}{b} + \frac{c+d}{e} - \frac{b+d}{f} + \frac{d+f}{b}$.
8. $\frac{a-b}{f} - \frac{a-c}{e} + \frac{a-d}{f} - \frac{b+d}{c}$.
9. $\frac{a+b}{d+e} + \frac{a-d}{e-f} - \frac{b-c}{a-d-e} + \frac{d-e}{b-e-f}$.
10. $4ae - \frac{15ad}{a+b-f} - 2cd - \frac{bc+ef}{be}$.
11. $abd - \frac{bce}{a+b+e-f} - \frac{e+f-c}{bdg} - def$.
12. $\frac{a+b-c+d-e+f}{a-b+c-d-e+f}$.

13. $\dfrac{a+d-e}{a-d-e} - \dfrac{a+cd-ef}{ad-def+e}$.

14. $\dfrac{a+b}{a-b} \times \dfrac{cd+d+e}{ae-ce} \div \dfrac{af+c}{b+c-d}$.

15. $\dfrac{ac}{b+d} \cdot \dfrac{a+d}{ef} \times \dfrac{ae}{df} \cdot \dfrac{bc}{a-e}$.

16. $\dfrac{9adf}{32be} \div \dfrac{8bce}{45af} \cdot \dfrac{15abf}{12cde} \div \dfrac{a+c-e}{a-c+f} \times \dfrac{b-d+e}{d-f+e}$.

17. $\dfrac{ac-bc+de}{cd-de-ef} - \dfrac{ad-ae-df}{bd-be-b} - \dfrac{ab-bf-def}{af+cf-cd}$.

18. $\dfrac{acd-bce-aef}{abf-bef-bdf-cf} - \dfrac{ade-cdf-ac}{cef-af-ef} - \dfrac{cde+de}{bd}$.

19. $\dfrac{5ad-3ab+5de}{3bc-7cf-2df} \times \dfrac{4ag+8bd}{3ae-4cf} - \dfrac{13de-11ef}{7ab-53ef}$.

20. $\dfrac{4adf-4cde+2d}{3bce-5ab-3ef} - \dfrac{3ad-bd}{2bc+3d} - \dfrac{7df-11c}{7e-2cf}$.

NOTATION—continued.

7. When two or more quantities are multiplied together the result is called their *product*, and the quantities so multiplied are called *factors* of the product. The product of more than two factors is called their *continued product*. If there be two factors only, each is called the *coefficient* (that is, co-factor) of the other; if one of them be a number, it is called the *numerical coefficient;* if a letter, the *literal coefficient*. When a quantity is broken up into as many factors as possible, these are called its *elementary factors*.

Thus, $12ab$ may be considered as the product of the two factors 12 and ab, where 12 is the numerical coefficient of ab, and ab the literal coefficient of 12; or as the continued product of 12, a, and b; or of the elementary factors 2, 2, 3, a, and b.

8. When all the factors are the same, their product is called a *power* of one of them. The product of *two* such factors is called the *second power* or *square* of one of them; of *three*, its *third power* or *cube;* of *four*, its *fourth power;* and so on. A quantity

is sometimes called the *first power* of itself. A small figure placed over a quantity, and to its right, indicates the power to which it is raised, or the number of factors of which it is composed, and is called the *index* or *exponent* of the power.

Thus, the product of the two factors, 5×5, called the square or second power of 5, 5 squared, or 5 to the second power, is written 5^2; the product of the three factors, $5 \times 5 \times 5$, called the cube or third power of 5, 5 cubed, or 5 to the third, is written 5^3; that of the four factors, $5 \times 5 \times 5 \times 5$, called the fourth power of 5, or 5 to the fourth, is written 5^4. $a \times a$, $a \times a \times a$, and $a \times a \times a \times a$, are read in the same way, and written a^2, a^3, a^4. a alone, that is, a to the first power, may be written a'. In every case, the index denotes the number of factors in the power.

9. When all the factors of a quantity are the same, one of them is called a *root* of the quantity.

The *square root* of a quantity is the factor whose square or second power gives the quantity; its *cube root*, the factor whose cube or third power gives the quantity; its *fourth root*, the factor whose fourth power gives it; and so on.

Thus, 5 is the square root of		25,	because	5^2, *i. e.* $5 \times 5 = 25$.
"	" cube	"	125,	"	5^3, *i. e.* $5 \times 5 \times 5 = 125$.
"	" fourth	"	625,	"	5^4, *i. e.* $5 \times 5 \times 5 \times 5 = 625$.
"	a^3 is the square	"	a^6,	"	a^6, *i. e.* $a^3 \times a^3 = a^6$.

The root of a quantity is indicated by putting the sign $\sqrt{}$ (a corruption of the letter *r* in *root*) before it; its square root by $\sqrt[2]{}$, or simply $\sqrt{}$; its cube root by $\sqrt[3]{}$; its fourth root by $\sqrt[4]{}$.

Thus, $\sqrt{25}$, $\sqrt[3]{27}$, mean the square root of 25, the cube root of 27. So $\sqrt{a^4}$ means the square root of a^4.

EXERCISE IV.

Read the following examples, using words for signs, and write them in algebraic form when read to you.

Find the values of

1. $6^4 - 9^2 - 8^3 + 4^5$.

2. $9 . 3^2 + 5 . 7^3 - 6 . 5^3 - 2^4 . 8^2$.

3. $\dfrac{3^4 \times 4^3}{2^5 \times 9^2} + \dfrac{5^4 - 15^2}{12^2 + 16^2} - \dfrac{3^5 + 5^2 - 11^2}{3^1 + 4^2 . 3^2}$.

4. $\sqrt{9} + 8\sqrt{16} - 2\sqrt{25} - 4\sqrt{1}$.

5. $4\sqrt[3]{125} - 6\sqrt[4]{81} - 2\sqrt[6]{1} + 3\sqrt[7]{128}$.

6. $\dfrac{\sqrt{4 \times 16}}{\sqrt{4} \times \sqrt{16}} - \dfrac{8\sqrt[3]{27}}{\sqrt[3]{8\sqrt[3]{27}}} + \dfrac{7\sqrt[5]{32} + 6\sqrt[4]{16}}{2\sqrt[3]{64} + 5\sqrt[9]{1}}$.

If $a = 4$, $b = 3$, $c = 2$, $d = 1$, $e = 0$, find the values of the following.

Example.

$$\frac{3^b - a^c}{b^e + 2d^e} + \frac{3\sqrt[2]{a}}{2\sqrt[4]{d}} = \frac{3^3 - 4^2}{3^2 + 2.1^4} + \frac{3\sqrt[2]{4}}{2\sqrt[4]{1}} = \frac{27 - 16}{9 + 2} + \frac{3 \cdot 2}{2 \cdot 1}$$

$$= \frac{11}{11} + \frac{6}{2} = 1 + 3 = 4.$$

7. $a^3 - b^3 - c^4 - d^6$.

8. $ab^2 - bc^3 - c^2d^4$.

9. $a^4b - bc^5 - b^3d^2$.

10. $a^2d^3 - b^2d^1 - c^6e^8$.

11. $3^2a - 3b^2 + 6^2c$.

12. $2^2a^2 + 2^3b^2 - 5^3d^2$.

13. $\dfrac{3a^2}{2c^3} - \dfrac{4b^3}{9c^2} + \dfrac{2a^3}{8cd^2}$.

14. $\dfrac{3^2 + a^4 - 3b^4}{a^2b^2 - 4c^5 - 5d^3} + \dfrac{10^2}{5^2c}$.

15. $a^a - b^ac^d + b^bc^e - e^d$.

16. $\dfrac{a^b - c^e}{a^eb} + \dfrac{ac + a^c}{c^b} - \dfrac{a^d}{c^e}$.

And, if $a = 64$, $b = 27$, $c = 16$, $d = 8$, $e = 4$, find the values of

17. $\sqrt{a} - \sqrt{c} - \sqrt[3]{d} - \sqrt{1}$.

18. $\sqrt[3]{a} - \sqrt[3]{b} + \sqrt[6]{a} - \sqrt[4]{1}$.

19. $3\sqrt[4]{c} + 2\sqrt[3]{a} - \sqrt[3]{b}\sqrt[6]{a}$.

20. $3\sqrt{c^2} - 4\sqrt[3]{d^3} - 2\sqrt[4]{e^4}$.

21. $\sqrt{\dfrac{a}{4}} + \sqrt{\dfrac{b}{3}} - \sqrt{\dfrac{a}{16}}$.

22. $\sqrt[3]{\dfrac{a}{d}} - \dfrac{\sqrt[3]{d}}{\sqrt{e}} + \sqrt{\dfrac{a}{c}}$.

NOTATION—continued.

10. *Brackets*, ( ), { }, [ ], are used to denote that all the quantities within them are to be treated as if they formed but one quantity.

Note the difference of meaning caused by the introduction of brackets in the following expressions. If $a = 16$, $b = 9$, $c = 2$:

$$a - b - c = 16 - 9 - 2 = 5;$$
$$a - (b - c) = 16 - (9 - 2) = 16 - 7 = 9.$$

$$a + bc = 16 + 18 = 34;$$
$$(a + b)c = (16 + 9) \times 2 = 25 \times 2 = 50.$$

$$bc^2 = 9 \times 2^2 = 9 \times 4 = 36;$$
$$(bc)^2 = (9 \times 2)^2 = 18^2 = 324.$$

$$\sqrt{a} + b = \sqrt{16} + 9 = 4 + 9 = 13;$$
$$\sqrt{(a + b)} = \sqrt{(16 + 9)} = \sqrt{25} = 5.$$

The *vinculum*, a line drawn over the quantities connected, is sometimes used instead of brackets. Thus, $a - (b - c)$, $\sqrt{(a + b)}$, &c., may be written $a - \overline{b - c}$, $\sqrt{\overline{a + b}}$, &c. In $\frac{a + b}{a - b}$ the line is really a vinculum, $\frac{a + b}{a - b}$ meaning the same as $(a + b) \div (a - b)$.

Exercise V.

Read the following examples, and write them down when read to you.

If $a = 6$, $b = 4$, $c = 3$, $d = 1$, $e = 0$, find the values of the following expressions.

Example.—Remove the inner brackets first, thus:

$$\{a^2 - (b^2 - c^2)\}\{c^b - c(b + d)^2\} + \sqrt[3]{ab + c}$$
$$= \{36 - (16 - 9)\}\{3^4 - 3(4 + 1)^2\} + \sqrt[3]{24 + 3}$$
$$= \{36 - 7\}\{81 - 75\} + \sqrt[3]{27} = (29 \times 6) + 3 = 174 + 3 = 177$$

1. $3a - b - c - d.$
   $3a - b - (c - d).$
   $3a - (b - c - d).$
   $3(a - b) - c - d.$

2. $a^2 - b^2 - c^2 - d^2.$
   $a^2 - b^2 - (c^2 - d^2).$
   $a^2 - (b^2 - c^2 - d^2)$
   $a^2 - (b^2 - c^2) - d^2.$

3. $3(a - b + c).$
   $3(a - b) + c.$
   $3a - (b + c).$
   $3a - b + c.$

4. $(a + b)(c - d).$
   $(a + b)c - d.$
   $a + b(c - d).$
   $a + bc - d.$

5. $(a + b)^3.$
   $a + b^3.$
   $(a + b + c)^2.$
   $a + b + c^2.$

6. $(a - d)^4.$
   $a - d^4.$
   $(a + b - c)^2.$
   $a + b - c^2$

7. $a^2b^2c^2$, $(abc)^2$, abc^2.

8. $\sqrt{a+c+b+c}$, $\sqrt{a+c}+b+c$, $\sqrt{b}+c+a+c$.

9. $\sqrt{24a-11b+23c}$, $\sqrt{24a-11b}+23c$, $\sqrt{24a}+23c-11b$.

10. $\sqrt{b(a+c)}$, $\sqrt{b}(a+c)$, $\sqrt{(a+c)b}$, $\sqrt{b}\sqrt{(a+c)}$.

11. $\sqrt{6ab}$, $\sqrt{a^2b}$, $\sqrt{a^2}b$, $\sqrt[3]{ab+c}$.

12. $\sqrt{4b^3}$, $\sqrt{4}b^3$, $\sqrt[4]{4b^3}$, $\sqrt[3]{4^2b^3}$.

13. $\sqrt{bc^2}$, $\sqrt{b}\sqrt{c^2}$, $\sqrt{b}c^2$, $\sqrt[a]{b^a}$.

14. $\sqrt{4(a+c)}$, $\sqrt{4}\sqrt{(a+c)}$, $\sqrt{4}(a+c)$,
$\sqrt{4^2(a+c)^2}$, $\sqrt{4^2}\sqrt{(a+c)^2}$, $\sqrt{4^2}(a+c)^2$.

15. $5a-2b-c-d-2b-(c-d)-(2b-c-d)$.

16. $5a-2b-(c-d)-\{2b-(c-d)\}-3de$.

17. $6e^2+2a^2-b^2-c^2+2a^2-(b-c)^2+(2a-b-c)^2$.

18. $5a^2-b^3-c^3-d^3-\{b^3-(c-d)^3\}-\{b-(c-d)\}^3$.

19. $\sqrt{16a^4}-\sqrt{a+b-d}-(a\sqrt[3]{a+c-d}\sqrt{25b})$.

20. $\{a-(b-c)\}\{a^2-(b-c)^3\}\{a-d^2(a-b)^2\}$.

21. $4a^2-[(b-c)\{a^2-(b-c)^3\}\{a-d^2(a-b)^2\}^2]$.

22. $2a^4-[a(b-c)a(b-d^2)^2(a-b)^2(c-d)]-e^2$.

NOTATION—continued.

11. The sign $\therefore$ means *then* or *therefore*; the sign $\because$, *since* or *because*.

12. The sign $\sim$, placed between two quantities, denotes their *difference*, and is used when we do not know which of them is the greater.

Thus, $a \sim b$ means either $a-b$ or $b-a$, according as a is greater than b, or b greater than a.

13. The sign $>$ means *greater than*; the sign $<$, *less than*.

Thus, $a > b$, means a is greater than b; $a < b$, means a is less than b.

14. The double signs, $\pm$ (read '*plus* or *minus*') and $\mp$ (read '*sum* or *difference*'), are sometimes placed between two quantities to denote their sum or difference, $\mp$ being used when it is not known which is the greater.

15. Quantities connected by *multiplication* are called *factors* (section 7); those connected by *addition* or *subtraction*, that is, by the signs $+$ or $-$, are called *terms*. *Simple* quantities consist of *one term*; *compound* quantities, of *more than one term*.

It is of great importance to distinguish factors from terms, and simple quantities from compound. A term may consist of a single quantity, as 12, a, b, $\sqrt{a^2}$; or of several factors. Thus, $12ab$ is a single term of three factors, 12, a, and b; $5ab \times 3cd$, a single term of two factors, $5ab$ and $3cd$; $4(a+b)(c+d-e)$, a single term of three factors, 4, $(a+b)$, and $(c+d-e)$, though $(a+b)$ is itself a compound quantity of two terms, and $(c+d-e)$ a compound quantity of three terms. 12, a, b, $\sqrt{a^2}$, a^4, $12ab$, $5ab \times 3cd$, $4(a+b)(c+d-e)$, are, therefore, simple quantities, and we see that a simple quantity may have a compound quantity for one of its factors. On the other hand, $4a + 5bc - d$ is a compound quantity of three terms, $4a$, $5bc$, and $-d$; $3(a+b) - c$, a compound quantity of two terms, $3(a+b)$ and $-c$.

16. A simple quantity is also called a *monomial*; a compound quantity of *two* terms, a *binomial*; of *three*, a *trinomial*; of *four*, a *quadrinomial*; of *more than four*, a *multinomial* or *polynomial*.

17. The order of the factors or of the terms that compose a quantity may be changed without affecting the result.

If $a = 3$, $b = 4$, $c = 5$, $2abc$, $2acb$, $2bca$, all $= 120$. A numerical coefficient, however, is always written first, and the letters in alphabetical order, unless for some special reason.

Similarly, $a - b + c$ will mean 3 things a person already has, 4 to be taken from him, and 5 to be given to him. We cannot begin by taking 4 from him, seeing he has only 3, but it will come to the same thing if we first give him 5, and then take away 4; so that $a - b + c = a + c - b = 4$. In the same way, $-b + a + c = a + c - b$, or $a - b + c$.

Exercise VI.

If $a = 6$, $b = 5$, $c = 4$, $d = 3$, $e = 2$, $f = 1$, find:

1. $(a + b - c) \sim (d - e + f)$.
2. $a + b - (c \sim d) - e + f$.
3. $adef \sim (a + d + e + f)$.
4. $a^2 \sim 2a$; $c^4 \sim cd$.

Shew by what quantity

5. $\sqrt[3]{b^2c} \angle \frac{b^2c}{e}$; $\sqrt[3]{2ce} \angle \frac{2ce}{c}$.

6. $cd^2 \angle (cd)^2$; $a(b+d) > (ab+d)$.

7. $(a+b)(c-e) > \{a+b(c-e)\}$.

8. $a^2c > ac^2$; $ab^2 \angle (ab)^2$.

9. Of how many terms does each example in Exercises II., III., IV., and V. consist? And how many factors has each of their terms?

10. Using only the signs of addition and multiplication, and keeping the letters in alphabetical order, combine a, b, c, d into the form of (1) a monomial, (2) a binomial, (3) a trinomial, in as many ways as you can.

ADDITION.

18. Quantities having + before them are called *positive* quantities; those having − before them, *negative* quantities. Similarly, + and − are called respectively the positive and negative signs, and sometimes simply *The signs*. When a quantity has neither + nor − before it, + is understood.

In $a-b+c-d$, a and c are positive quantities; $-b$ and $-d$ negative quantities.

19. When a quantity has no numerical coefficient expressed, 1 is understood. Thus, a means $1a$; that is, once a.

20. Quantities that are exactly alike except in their signs and coefficients are called *like* quantities; other quantities are called *unlike*.

Thus, a, $5a$, $-7a$ are like quantities; so are a^2b^3, $4a^2b^3$, $-5a^2b^3$; $-bx^2y$, $5bx^2y$, $-8bx^2y$. But ab, cd, ef are unlike quantities; so are a^2, a^3, a^4; $7a$, $5ab$, $2abc$; and a^2bc, ab^2c, and abc^2.

Such quantities as ac^2d, $-bc^2d$, and ec^2d may be, and often are, considered like quantities, for they differ only in their literal coefficients, a, $-b$, and e. Similarly, $abxy$, $3cdxy$, and $-5xy$ may be considered like, as they differ only in their coefficients ab, $3cd$, and -5.

21. In algebra, as in arithmetic, only like quantities can be added together or subtracted one from another.

If required to add 1 pound, 6 shillings, and 4 pence, all we can do is to write down £1, 6s. 4d., till we change them into the like quantities 240d. + 72d. + 4d. = 316d.; if required to subtract 6 shillings from 1 pound, we can merely write £1 − 6s., or change them into the like quantities, 20s. − 6s. = 14s. Similarly, $4a$, $5a$, $6a$ added together, give $15a$; $4a$, $5b$, $6c$ added together, can merely be written $4a + 5b + 6c$; $4ab$ subtracted from $6ab$ gives $2ab$; $4cd$ subtracted from $6ab$ can merely be written $6ab - 4cd$.

Case I.—To Add Like Quantities.

22. Rule.—Find the difference between the sum of the positive and the sum of the negative coefficients, prefix the sign of the greater, and annex the common letter or letters.

Examples.

(1.)	(2.)	(3.)	(4.)
£9	9s.	$9a$	$9a$
£1	1s.	a	$-a$
£7	7s.	$7a$	$-7a$
£17	17s.	$17a$	a

(5.)	(6.)	(7.)	(8.)
$-9a$	a	$6ab$	$5a^2x$
$-a$	$7a$	$7ab$	$-17a^2x$
$7a$	$-9a$	$-8ab$	$9a^2x$
$-3a$	$-a$	$5ab$	$-3a^2x$

In these examples, where no sign is given, + is understood; and a and $-a$, standing alone, mean $+1a$, $-1a$ (sect. 19). We do not add the a's together, just as we do not add the £'s and s's.

Let a mean a bag containing 4 marbles; then, in (3), $9a + a + 7a$ will mean 9 bags, 1 bag, and 7 bags, altogether $17a$; that is, 17 bags, to be added to a boy's stock. In (4), $9a$, $-a$, and $-7a$, will mean 9 bags to be added to it, and 1 bag and 7 bags to be taken from it, altogether 9 bags to be added to it, and 8 bags to be taken from it; and the answer, a, that is, $+a$, means that all this giving and taking simply amounts to at once giving him 1 bag. In (5) and (6), the answers, $-3a$ and $-a$, mean that all the giving and taking in these examples simply amounts to at once taking 3 bags, or 1 bag, from him. In (8), if $a = 3$, $x = 2$, a^2x may be taken to mean a bag containing 18 marbles. The same results will be arrived at, if we take *positive* quantities to

mean money a person has, or *property*; and *negative* quantities, money he has to give away, or *debt*.

EXERCISE VII.

1. $3a + 2a + 4a + 5a + 8a + 6a$.
2. $6a + 7a + a + 12a + 9a + 4a$.
3. $4ab + 6ab + 7ab + ab + 3ab$.
4. $3a^2b + 5a^2b + a^2b + 2a^2b$.
5. $-3x - 4x - x - 8x - 9x - 6x$.
6. $-4y - 5y - y - 8y - 7y - 6y$.
7. $-6ax - 9ax - 3ax - ax$.
8. $-5ab^2 - 4ab^2 - ab^2 - 7ab^2$.
9. $13a + 9a - 3a - 4a - 11a$.
10. $5ab + 7ab - ab - 6ab + 3ab$.
11. $12xy - xy - 7xy + 4xy - 5xy$.
12. $8x^2y - 7x^2y - 4x^2y + 6x^2y$.
13. $15ax - 9ax + ax - 8ax + 2ax$.
14. $9xy - 7xy - 3xy - xy + 2xy$.
15. $13b^2c - 9b^2c + b^2c - 4b^2c$.
16. $11ax^3 - ax^3 + 7ax^3 - 18ax^3$.
17. $5ac - 6ac + 4ac + 7ac - 9ac$.
18. $7abc - 9abc - 5abc + 4abc$.
19. $-7xyz + 9xyz + 6xyz - 8xyz$.
20. $8cd^3 - 13cd^3 - 5cd^3 + 9cd^3$.
21. $-8ax^2 - 7ax^2 + 12ax^2 + 6ax^2$.
22. $4xz^2 - 8xz^2 - 2xz^2 + 7xz^2$.
23. $9cd - 5cd + cd + 2cd - 8cd$.
24. $7d - 9d - 8d + 5d - 6d + 11d$.
25. $-6xz - xz + 9xz - 8xz + 4xz$.
26. $-7ay + 3ay + ay - 5ay + 8ay$.
27. $-5ay^2 + 9ay^2 - ay^2 + 7ay^2$.
28. $-4x - 6x + x - 7x + 10x$.
29. $6ax^2y - 7ax^2y + 13ax^2y + 19ax^2y - 9ax^2y + 15ax^2y - ax^2y$.
30. $-14a^2x^2 + 9a^2x^2 - 17a^2x^2 + 5a^2x^2 - a^2x^2 - 8a^2x^2 - a^2x^2$.
31. $8abc + 9abc - 7abc - 6abc + 5abc - 4abc + 10abc - 14abc$.
32. $-a^3b^2 + 8a^3b^2 + 3a^3b^2 - a^3b^2 - 7a^3b^2 + 9a^3b^2 - 11a^3b^2$.
33. $15bz^2 - 7bz^2 - bz^2 + 8bz^2 - 9bz^2 - 15bz^2 + 6bz^2 + 2bz^2$.
34. $9a^2x + 7a^2x - a^2x + 3a^2x - 4a^2x + 8a^2x - 29a^2x + 6a^2x$.
35. $-7ax^2y + 4ax^2y - 8ax^2y + ax^2y - 9ax^2y + ax^2y - 6ax^2y$.
36. $-15x^2yz + 18x^2yz - 17x^2yz + 19x^2yz + 10x^2yz - 14x^2yz$.

37. Work each of these exercises again, changing every sign.

38. Choose some number (greater than 0) to stand for each letter of each sum; find the value of each term, and add these values together; if the result be the same as the value of the answer, the answer will be correct.

20. Why should each number chosen be greater than 0?

CASE II.—TO ADD QUANTITIES PARTLY LIKE AND PARTLY UNLIKE.

23. RULE.—Add each set of like quantities as in Case I.

Examples.

(1.) $4a + 5a^2b - 6a - 8c - 7a^2b + a^2b + e + c - a + 5c.$

(2.) $5(a + x) + 5bc - (a + x) - 6ax - b - 3bc + ax + 4(a + x) + 5ax.$

$$\begin{array}{llll}
(1.) & 4a + 5a^2b - 8c + e & (2.) & 5(a + x) - 6ax + 5bc - b \\
 & -6a - 7a^2b + c & & -(a + x) + ax - 3bc \\
 & -a + a^2b + 5c & & 4(a + x) + 5ax \\
\hline
 & -3a - a^2b - 2c + e & & 8(a + x) \qquad + 2bc - b
\end{array}$$

At first it is better to arrange the like quantities in columns; after some practice, the pupil will be able to pick them out and add them by merely glancing along the line. Care must be taken to give each part of the answer its proper sign; in (2), $-b$, without a sign, would mean $+b$. The answer is the same with whatever column we begin, but in algebra it is more convenient to work from left to right.

EXERCISE VIII.

Find the sum of

1. $3a + 4b - 2c,\ -5a + 3b - 7c,\ 6a - 2b + 6c,\ -a + 5b - 3c.$

2. $-4a - 5b - c + 6,\ 4b - 7c - 7,\ -2a + 3b + 9c,\ -b - 8.$

3. $5x - 3y + 2z,\ 4y + 8,\ -3x - 7z - 6,\ 2x + 8y + z.$

4. $a^3 + a^2b - 3ab^2,\ -7a^3 - 8a^2b - ab^2,\ 4a^3 + 9a^2b + 8ab^2,\ 8a^3 + a^2b - 5ab^2,\ -9a^3 - 7a^2b + 6ab^2,\ 2a^3 + 3a^2b - 7ab^2.$

5. $3ax^2 - 5xy^2 - 14cz^2 + 8,\ -2ax^2 + 6xy^2 - 3cz^2,\ 7xy^2 + 18cz^2 - 5,\ 5ax^2 - 6cz^2 - 6,\ -10ax^2 - 9xy^2 + 3,\ xy^2 + 4cz^2.$

6. $-7ab^3 + 5ab^2 - 3ab + 3a,\ 6ab^3 - 4ab^2 - 5a,\ -9ab^3 + 6ab^2 - 4ab,\ 3ab^2 + 6ab - 7a,\ 8ab^3 - 5ab + 6a,\ 2ab^3 - 9ab^2 + 6ab.$

7. $-3a^2xy + 4ax^2y - 2axy^2 + 4a^2xy - ax^2y + 7axy^2 - 9a^2xy - 8axy^2 + 6ax^2y - 8ax^2y + 3axy^2 - 2a^2xy - axy^2.$

8. $ax^3 - 4ax^2 - 5x^3y + 8x^2z - 3ax^3 - 5ax^2 + 4x^3y + 4ax^2 + 6ax^3 - 6x^2z - 7ax^2 - 3x^3y + 8x^2z - ax^3 + 4x^3y + 5x^2z.$

9. $8c^2d^2 - 4d^2e^2 + 5xy + 5xz + 7c^2d^2 + 5d^2e^2 - 8xy - 7d^2e^2 + 4xy - 3xz - 6c^2d^2 - 5xy - 4xz - 5c^2d^2 + 6d^2e^2 + 3xy.$

10. $6a^2 - a^2x + 7a^2y - b^2 + by - 3a^2 + 2a^2x - 5a^2y + 8b^2 - 9by - 4a^2x - 6b^2 - 5by - 4a^2 - 2a^2y + 13by + 6a^2x - 2b^2.$

11. $-3a^2x + 4a^2xy - 5b + 3c - 6a^2xy + 7b - 5c + 5a^2x - 9b + 7c - 7a^2x + 8a^2xy - c + 5a^2x - 3a^2xy + 6b - 4a^2xy - 4c.$

12. $8x^2yz - 3xy^2z + 3xyz^2 - 3xyz - x^2yz + 7xy^2z - 6xyz^2 + 5xyz + 8x^2yz - 5xy^2z - 4xyz^2 - 8xyz - 12x^2yz + 7xyz^2.$

13. $-a^2b - 7a^2c - 7ab^2 + 8a^2c - 8a^2b + 6a^2b + 8ab^2 + ac^2 + 2ac^2 + 2a^2b - 5a^2c + 6ac^2 + 4a^2c - 3ab^2 - 9ac^2 + ab^2.$

14. $-7ae + 6ax - 4ae + 2ax$, $-4ax + 8ay + 3ae + 3ay$, $-5ay - 7xy - 7yz - 2xy$, $3xy + 4yz - 6ay + 8ae$, $-6yz + 9yz + 6xy - 5ax.$

15. $5x^2y^2 - 4x^2y + 3xy^2 - 2xy$, $8x^2y + 5xy - 7x^2y^2 - 6xy^2$, $5xy^2 - 2x^2y + 9x^2y^2 - 7xy$, $6xy - 2xy^2 - 3x^2y - 8x^2y^2.$

16. $3x^3y - 4x^2y^2 - 2x^3z - 5xz^3$, $3x^2y^2 + 5xy^3 - 4x^2z^2 + 3xz^3$, $-4x^3y - 4xy^3 + 3x^2z^2 + x^2y^2 + x^3z + 2xz^3$, $2x^3y + xy^3.$

17. $a^6 - a^4b^2 + a^2b^4 - b^6$, $a^5b + a^3b^3 - ab^5$, $2a^4b^2 - 3a^2b^4 - a^6 + 4b^6$, $2ab^5 - 8a^3b^3 - 8a^5b - 3b^6$, $a^3b^3 - ab^5 + 2a^2b^4 - b^6 - a^6.$

18. $x^7 + x^2y^5 - x^4y^3$, $3x^3y^4 - 2xy^6 + x^6y$, $3x^4y^3 - 3x^7 - z^2y$, $2x^6y + zy^2 + 2x^3y^4 + 8xy^6$, $3x^2y^5 - 4z^2y + 5x^4y^3 - 4x^3y^4 + 4x^7$, $3z^2y - 4x^2y^5 - 6x^4y^3 - 2zy^2 - 3x^7.$

19. $4a^3b - 5a^2b^2 + 6ab^3$, $3a^3b + 7a^2b^2 - 9ab^3$, $-5a^3b - 6a^2b^2 + 5ab^3$, $7a^3b + 6a^2b^2 - 3ab^3$, $-6a^3b - a^2b^2 + 4ab^3.$

20. $3a^2b^2 - 2a^2b + 4ab - 5ab^2$, $a^2b - 5a^2b^2 - 2ab + 7ab^2$, $-6a^2b^2 - 5a^2b + 3ab + 8ab^2$, $a^2b^2 - 4a^2b + ab - 9ab^2.$

21. $7a^2xy - 5ax^2y - 4axy^2 - 6axy$, $-9a^2xy + 7ax^2y + 6axy^2 + 4axy$, $5a^2xy - 9ax^2y - 8axy^2$, $-2a^2xy + 6ax^2y + 5axy^2 + axy.$

22. $-8ax^2 + 2bx^2 - 4cx^2 + 5dx^2$, $ax^2 - 6bx^2 - 5cx^2 - 3dx^2$, $2ax^2 - 3bx^2 + 6cx^2 - 7dx^2$, $ax^2 + 5bx^2 + 2cx^2 + 4dx^2.$

23. $4a^2b^2x - 5a^2bx - 6ab^2x - 3abx$, $2a^2bx - 7a^2b^2x + 4ab^2x + 5abx$, $3a^2b^2x - 4a^2bx - 2ab^2x - abx$, $3ab^2x + 6a^2bx - a^2b^2x.$

24. $3x^2y - 4x^2y^2 + 5xy^3 - 4x^2z$, $5x^2z - 5x^2y + 7x^2y^2 - 9xy^3 + 4x^2z$, $-3xy^3 - 5x^2y^2 - 2x^2y$, $3x^2y - 6x^2z + 6xy^3 + x^2y^2$.

25. $4x(a-b) - 5y(c-d)$, $-x(a-b) - 3y(c-d)$, $3x(a-b) + y(c-d)$, $-2x(a-b) + 6y(c-d)$.

26. $7x(a-b) + (c-d)y + xy$, $x(a-b) - 7(c-d)y + mn$, $-4x(a-b) + 3(c-d)y - 2mn$, $-3x(a-b) + 2(c-d)y$.

27. $-3(a-b)x + xy - 6(m-n)z$, $(a-b)x + 4(m-n)z$, $-4(a-b)x - 2xy - (m-n)z$, $5(a-b)x + 2(m-n)z$.

SUBTRACTION.

24. The quantity from which another is subtracted is called the *minuend*; the quantity that is subtracted, the *subtrahend*; the result, the *remainder* or *difference* (L. *minuendum*, to be lessened; *subtrahendum*, to be subtracted).

25. RULE.—Change the signs of the terms in the *subtrahend*, and proceed as in addition.

Examples.

(1.) $\begin{array}{r} 9 \\ 7 \\ \hline 2 \end{array}$	(2.) $\begin{array}{r} 9 \\ -7 \\ \hline 16 \end{array}$	(3.) $\begin{array}{r} -9 \\ 7 \\ \hline -16 \end{array}$	(4.) $\begin{array}{r} -9 \\ -7 \\ \hline -2 \end{array}$
(5.) $\begin{array}{r} a \\ b \\ \hline a-b \end{array}$	(6.) $\begin{array}{r} a \\ -b \\ \hline a+b \end{array}$	(7.) $\begin{array}{r} -a \\ b \\ \hline -a-b \end{array}$	(8.) $\begin{array}{r} -a \\ -b \\ \hline -a+b \end{array}$

If John has 9 marbles, but owes James 7, his stock, after James is paid, is evidently $9 - 7 = 2$. Thus we see that the 7 to be subtracted is changed to -7, and -7 then added to 9. Therefore, 7 subtracted $= -7$ added. Similarly, b *subtracted* from a is the same as $-b$ *added* to a; that is, $a - b$.

John's stock will be increased to the same extent whether we add 7 to it with which to pay James, or pay James for him, and thus take away his debt; for, in either case, it will, after James is paid, be 9. Thus, to *take away a debt of* 7 is the same as to *add* 7, that is, -7 *subtracted* $= 7$ *added*, or $-(-7) = 7$. Similarly, $-b$ *subtracted* from a, $= b$ added to a, $= a + b$.

In (1), 7 subtracted from 9, $= -7$ added to 9, $= 9 - 7 = 2$; in (2), -7 subtracted from 9, $= 7$ added to 9, $= 9 + 7 = 16$; in

(3), 7 subtracted from -9, $= -7$ added to -9, $= -9 - 7 = -16$; in (4), -7 subtracted from -9, $= 7$ added to -9, $= -9 + 7 = -2$.

In (5), (6), (7), and (8), it is seen that the minuends retain their own signs in the answer, while those of the subtrahends are all changed; and that the results cannot be expressed as single quantities, but only as $a - b$, $a + b$, $-a - b$, $-a + b$ (sect. 21). They differ from (1), (2), (3), and (4), only in having letters for figures.

It is better at first actually to change the signs, and to arrange like quantities under like; after some practice, the pupil may merely suppose the signs changed, and proceed without arranging.

Exercise IX.

In the following exercises the minuend comes first, separated from the subtrahend by a comma.

1. (1.) $7a$, $4a$; (2.) $6a^2$, $5a^2$; (3.) $8ab$, $9ab$; (4.) $5x^2$, $-3x^2$; (5.) $7b^2$, $-b^2$; (6.) $11xy$, $-12xy$; (7.) $-5x$, $4x$; (8.) $-7x$, $6x$; (9.) $-3b^2$, $4b^2$; (10.) $-8a$, $-7a$; (11.) $-5y^2$, $-6y^2$; (12.) $-6z$, $-6z$; (13.) ax, ax; (14.) ax, $-ax$; (15.) $-ax$, ax; (16.) $-ax$, $-ax$; (17.) $3ab$, $-3ab$; (18.) $-3ab$, $3ab$; (19.) $-3ab$, $-3ab$.

2. (1.) $5ab + 6ac$, $3ab + 4ac$; (2.) $13xy + 15xz$, $10xy + 12xz$, (3.) $18a^2x + 17b^2x$, $14a^2x + 13b^2x$.

3. (1.) $9a^2c + 8ac^2$, $4a^2c + 5ac^2$; (2.) $4x^2y + 7x^2z$, $3x^2y + 6x^2z$; (3.) $11x^2y + 10xy^2$, $5x^2y + 4xy^2$.

4. (1.) $12x^3y^2 + 15x^2y^3$, $5x^3y^2 + 7x^2y^3$; (2.) $24axy + 36bxy$, $9axy + 18bxy$; (3.) $8a^3x + 7a^2x^2 + 9ax^3$, $4a^2x^2 + 5a^3x + 6ax^3$.

5. (1.) $6ad + 5bd$, $-3ad - 4bd$; (2.) $7a^2d + 4ad^2$, $-5a^2d - 3ad^2$; (3.) $6a^3y + 5ay^3$, $-3a^3y - 7ay^3$.

6. (1.) $-7xy - 8xz$, $2xy + xz$; (2.) $-8a^3x^2 - 9a^2x^3$, $2a^3x^2 + 4a^2x^3$; (3.) $-5x^2yz - 4xy^2z - 3xyz^2$, $3x^2yz + 6xy^2z + xyz^2$.

7. (1.) $-5ab - 3bc$, $-4ab - 2bc$; (2.) $-9ac^2 - 10c^2d$, $-5ac^2 - 4c^2d$; (3.) $-9x^2y - 2xy - 4xy^2$, $-3x^2y - 6xy - 6xy^2$.

8. $5x^4y - 6x^3y^2 + 4x^2y^3 - 3xy^4$, $3x^4y + 4x^3y^2 - 2x^2y^3 + 3xy^4$.

9. $-a^2bc - ab^2c + abc^2 - abc$, $a^2bc + ab^2c - abc^2 + abc$.

10. $3ab - 4bc - 3cd + 5de + 6ef,\ 4ab + 5bc - 2cd + 5de - ef.$

11. $x^4 - x^3y + x^2y^2 - xy^3 - y^4,\ -x^4 - x^3y - x^2y^2 + xy^3 - y^4.$

12. $-x^5 - 3x^4 + 4x^3 + 5x^2 - 4x - 3,\ x^5 - 2x^4 + 3x^3 - 2x^2 - 5x + 2.$

13. $-a^5 + a^4 - a^3 - 3a^2 + 4a + 3,\ -a^5 - a^4 - 3a^3 + a^2 + 5a + 3.$

14. $7x^5 - 6x^4 + 3x^3 + 4x^2 - x - 2,\ -6x^5 + 6x^4 + 3x^3 - 4x^2 - x + 2.$

15. $5x(a - b) + 7(c - d)y,\ 3x(a - b) + 4(c - d)y.$

16. $-2(a - b)x + 3(c - d)y,\ 3(a - b)x - (c - d)y.$

17. $-(a - b)x^2 - (c - d)x,\ (a - b)x^2 + (c - d)x.$

18. $(m - n)x^2 - (n + p)x,\ -(m - n)x^2 - (n + p)x.$

19. From $3a^2b - 5ab + 6ab^2$, take $-3a^2b + 5ab - 2ab^2$ and $-4a^2b - 10ab + 3ab^2$.

20. From $a^4b - 7a^3b^2 + 3a^2b^3 - ab^4$, take $-5a^4b - 3a^3b^2 - 6a^2b^3 - 2ab^4$ and $6a^4b + 4a^3b^2 + a^2b^3 - 3ab^4$.

21. Take $x^5y - 2x^4y^2 + 4x^3y^3 - 2x^2y^4 - xy^5$ and $4x^5y + 2x^4y^2 - x^3y^3 - 4x^2y^4 + 4xy^5$, from $-x^5y + 2x^4y^2 - 3x^3y^3 - 2x^2y^4 + xy^5$.

BRACKETS.

26. *Resolution of Brackets.*—(1.) The sign $+$ before a bracket means that every quantity within the bracket is to be added (sect. 10). Therefore, $+(b + c - d) = +b + c - d$; where we see that, *when $+$ precedes a bracket, we may remove the $+$ and the bracket if we retain the sign of every quantity within the bracket.*

(2.) The sign $-$ before a bracket means that every quantity within the bracket is to be subtracted. To subtract quantities, we change their signs and add. Therefore, $-(b + c - d)$ means $b + c - d$ to be subtracted, which is the same as $-b - c + d$ to be added; where we see that, *when $-$ precedes a bracket, we may remove the $-$ and the bracket if we change the sign of every quantity within the bracket.* Thus,

$$a+(x-y+z) = a+x-y+z;\quad a+(-x-y+z) = a-x-y+z.$$
$$a-(x-y+z) = a-x+y-z;\quad a-(-x-y+z) = a+x+y-z.$$

Where bracket occurs within bracket, it is safer to remove the inner bracket first. After this method is mastered, the pupil may try the plan of removing the outer bracket first, and use it as a test of the accuracy of his work by the former method. But it

must be carefully remembered that the sign before an outer bracket does not affect any quantity within the inner bracket. Thus,

$$a - [b - \{c - (a - b + c) + (a - b + c) - a\} - (a + b - c)]$$

By first method:

$$= a - [b - \{c - a + b - c + a - b + c - a\} - a - b + c]$$
$$= a - [b - c + a - b + c - a + b - c + a - a - b + c]$$
$$= a - b + c - a + b - c + a - b + c - a + a + b - c = a$$

By second method:

$$= a - b + \{c - (a - b + c) + (a - b + c) - a\} + (a + b - c)$$
$$= a - b + c - (a - b + c) + (a - b + c) - a + (a + b - c)$$
$$= a - b + c - a + b - c + a - b + c - a + a + b - c = a.$$

Exercise X.

Reduce to their simplest forms, by clearing of brackets:

1. $a + (a - b)$; $a + (-a + b)$; $a + (a + b)$.

2. $a - (a + b)$; $a - (a - b)$; $a - (-a + b)$; $a - (-a - b)$.

3. $a - b + c + (a + b - c)$; $a + b - c - (a + b - c)$.

4. $a - x + (a - x) - (a - x)$; $a + x - (a + x) - (a - x)$

5. $a - b - c - (a - 2b + c) - (2a + b - 2c)$; $a - b - c - (a - b + c) - (a - b - c)$; $a - b + c - (a + b - c) - (a - b + c)$.

6. $2a - 2b - 2c - (3a + 3b - 3c) - (4a - 4b + 5c) - (-6a - b - 4c)$.

7. $a - b + c - (3a + 4b - 5c) - (2a - 5b + 6c) - (b - c - 4a)$.

8. $-(3x + 4y - 5z) + (4x + 3y - 4z) - (5x + 2y - 2z) - (5y - 6z - 9x) - (6x - 7y - 4z) + (3x + 2y - 7z) - (5x + 6y + 3z) - (3z - 2x - 4y)$.

9. $a - \{a - b - (a - b)\}$; $a - \{a + b - (a + b)\}$.

10. $a - \{a - b + (a - b) - (a - b)\}$; $a - \{a + b - (a - b) - (a + b)\}$.

11. $6a^2 - (4ab - b^2) - \{3a^2 + (2ab + 5b^2)\} - \{2a^2 - (6ab + 3b^2)\}$.

12. $3a^2 - \{2ax - (4a^2 + 5ax)\} - \{a^2 + (3ax - x^2) - (a^2 - x^2)\}$.

13. $1 - \{1 - (1 - 2)\} + \{1 - (2 - 1) - (1 - 2)\} - \{1 - (2 - 1)\}$.

14. $5a - \{b + (c - a) - (2b + c)\} - \{4a - (c - b)\} - \{a - (b - c)\}$.

15. $4x^2 - [xy - \{y^2 - 3x^2 - (2x^2 - xy) + xy - (xy - y^2)\} - x^2]$.

16. $x^3 - [x^2y - \{xy^2 - (y^3 + x^3)\} - x^2y] - (xy^2 + y^3) - [x^3 + \{x^2y - (xy^2 + y^3)\}]$.

17. $1 - [1 - \{1 - (1+2) - (1-2)\} - \{1 + (1-2)\}] - [1 + \{1 - (0 - 1)\}]$.

18. $3a - [2b - \{3c - (3a - 2b + 3c) + (2a - 3b)\} + a - \{3b - 2c - (3b - 2c)\}] - [5a - 3c - \{5a + (3b - 4c)\} + \{a - (b + c)\}]$.

27. *Formation of Brackets.* — When quantities are placed within a bracket, *the bracket will have + before it, if the signs of the quantities placed within it are not changed; but − before it, if they are changed.* Thus,

$a + x - y + z = a + (x - y + z)$; or, putting y first, $a - (y - x - z)$;

$a - x - y + z = a + (-x - y + z)$; or rather, $a - (x + y - z)$, for the form $a + (-x - y + z)$ is very seldom written.

With a double bracket:

$a + x - y + z = a + \{x - (y - z)\}$, or $a - \{y - (x + z)\}$.

$a - x - y + z = a - \{x + (y - z)\}$, or $a - \{x - (z - y)\}$.

Great care should be taken in forming a double bracket; after it is formed, its accuracy may be tested by resolving it again.

Exercise XI.

Express by brackets, keeping the terms in the order given, and taking them (1) *two*, (2) *three*, together.

1. $a - b + c - d + e - f$. 2. $a + b - c + d - e + f$.

3. $-a + b - c - d - e - f$. 4. $-f + e - d - b - c + a$.

5. $3a + 2b - 4c - x - y + z$. 6. $-3a + 2b + 4c - x - y - z$.

7. $-3a - 3b - 4c - x + y - z$. 8. $-4c + 3b - 3a - z - y + x$.

9–16. Express the above by enclosing the last *four* terms in each in an *outer* bracket, which shall include the last *three* in an *inner* bracket.

17–24. Express them also by enclosing the *first three* terms in an *outer* bracket, which shall include the *second and third* in an *inner* bracket; and the *last three* in an *outer* bracket, which shall include the *fifth and sixth* in an inner.

MULTIPLICATION.

28. The quantity *to be multiplied* is called the *multiplicand*; the quantity *we multiply by*, the *multiplier*; the result, the *product*.

29. Revise sections 7, 8, 18, and 19, and carefully note that,

1. Quantities may be multiplied in any order.

Thus, $3ac \times 4bd = 3 . a . c . 4 . b . d = 3 . 4 . a . b . c . d = 12abcd$; and $2ad \times 3be \times 5cf = 2 . 3 . 5 . a . b . c . d . e . f = 30abcdef$. The letters may therefore be written in alphabetical order.

2. Rule of Signs.—*Like signs* give *plus*; *unlike signs* give *minus*.

Let $a = 6$, $b = 4$. Then $-a = -6$, $-b = -4$; and

(1.) $a \times b$ will mean, say, 6 marbles given to a boy 4 times; that is, 24 marbles given to him. So that $6 \times 4 = 24$, or $a \times b = ab$, where we see that a *positive* quantity multiplied by another *positive* quantity gives a *positive* product.

(2.) $-a \times b$ will mean *a debt* of 6 marbles put to his stock 4 times; that is, a whole *debt* of 24 put to it, or -24 put to it. So that $-6 \times 4 = -24$, or $-a \times b = -ab$, where we see that a *negative* quantity multiplied by a *positive* quantity gives a *negative* product.

(3.) $a \times -b$ will mean 6 marbles *taken away* from him 4 times; that is, 24 *taken away* from him, which (sect. 25) is the same as putting -24 to his stock. So that $6 \times -4 = -24$, or $a \times -b = -ab$, where we see that a *positive* quantity multiplied by a *negative* quantity gives a *negative* product.

(4.) $-a \times -b$ will mean *a debt* of 6 marbles *taken* from him 4 times; that is, a whole *debt* of 24 from which he is to be relieved, which (sect. 25) is the same as adding 24 to his stock. So that $-6 \times -4 = 24$, or $-a \times -b = ab$, where we see that

a *negative* quantity multiplied by a *negative* quantity gives a *positive* product.

From (1) and (4), we see that *two positive* or *two negative* quantities multiplied together give a positive product; and from (2) and (3), that a *positive* quantity and a *negative* quantity multiplied together give a *negative* product. Hence the *rule of signs*, as given above. In practice, it is best to apply it thus: *a positive multiplier leaves the signs of the multiplicand unaltered; a negative multiplier changes them* (*plus* to *minus*, *minus* to *plus*).

Thus, $3a \times 4b = 12ab$; $-2ac \times 3bd = -6abcd$; $3ab \times -4c = -12abc$; $-4ax \times -5by = 20abxy$.

3. Rule of Indices.—To multiply powers of a quantity together, *add their indices*.

$$a = a^1;\ a \times a = a^2;\ b \times b = b^2;\ \text{\&c. (sect. 8)}.$$

$$a \times a^2 = a^{1+2} = a^3;\ \because a \times a^2 = a \times a \,.\, a = a \,.\, a \,.\, a = a^3.$$

$$a^2 \times a^2 = a^{2+2} = a^4;\ \because a^2 \times a^2 = a \,.\, a \times a \,.\, a = a \,.\, a \,.\, a \,.\, a = a^4.$$

$$a^2 \times a^3 = a^{2+3} = a^5;\ \because a^2 \times a^3 = a \,.\, a \times a \,.\, a \,.\, a = a \,.\, a \,.\, a \,.\, a \,.\, a = a^5.$$

Similarly, $7ab \times 3a^2b = 21a^3b^2$; $5a^2b^3 \times -4ab^3 = -20a^3b^6$; $3abc \times -4ab^2 \times 2ab^5y = -24a^3b^8cy$.

Case I.—To Multiply Simple Quantities together.

30. Rule.—Multiply the coefficients together for the coefficient of the product; write the letters after it in alphabetical order; and observe the rule of signs and the rule of indices.

Examples.—See the two preceding sections.

Exercise XII.

1. $3ab \times 4cd$; $5ab \times 2cx$; $5ac \times 3dy$; $4ad \times 2cx$.
2. $4abx \times 3cdy$; $9aey \times 4bcd$; $12aby \times 9cex$.
3. $-6ab \times 2cd$; $-4ac \times 5be$; $-7bd \times 3ae$.
4. $-5ab \times 3cd$; $-6bcd \times 7aey$; $-8aby \times 9cx$; $-7ax \times 5by$.
5. $3ab \times -7cd$; $8ac \times -7bd$; $6ax \times -9by$; $9acy \times -8bdx$.
6. $-9a \times -7b$; $-8x \times -9y$; $-8a \times -9b$; $-7y \times -8ax$.
7. $ab \times cd$; $-ax \times y$; $-ad \times -bc$; $-ax \times bd$; $-axz \times -by$.
8. $7ab \times -xy$; $ax \times -11by$; $-9az \times -xy$; $-15by \times -7xz$.

9. $8ac \times -9bd$; $-7ax \times 6by$; $-6ay \times -4x$; $-7x \times -3ay$.

10. $-9x \times 5ay$; $-4y \times -7ax$; $4ad \times -9bc$; $-8ac \times -4bx$.

11. Find the squares of (1.) a, $2a$, $3a$, $7a$; (2.) ab, $2ab$, $3ab$, $5ab$; (3.) abc, $2abc$, $3abc$, $6abc$; (4.) $-a$, $-2a$, $-3a$, $-4a$; (5.) $-ab$, $-2ab$, $-3ab$, $-9ab$; (6.) $-abc$, $-2abc$, $-3abc$.

12. Find the cubes of the same quantities.

13. (1.) $a \times a^2$, $a \times a^3$, $-a \times a^4$; (2.) $a \times -a^2$, $-a \times -a^5$, $-a \times -a^6$; (3.) $ab \times a^2b^2$, $ab \times a^3b^3$, $-ab \times a^5b^4$, $-ab \times a^6b^6$; (4.) $ab \times -a^3b^3$, $-ab \times -a^5b^6$, $-ab \times -a^4b^4$.

14. (1.) $a^2b^2 \times a^2b^2$, $a^2b^2 \times a^3b^3$, $-a^2b^2 \times a^5b^4$, $-a^2b^2 \times -a^3b^2$; (2.) $a^2x \times ax^2$, $-a^3x \times ax^2$, $-ax^2y \times -a^2xy$, $-a^3x^2y \times -ax^2y^3$.

15. $7a^2x \times -9ax^2$; $-5ax^3 \times -9ax$; $-17ay^2 \times a^2xy$; $ax \times -a^2xz$.

16. $-ax^2y \times -3a^2xy$; $-5abx \times 7a^4b^3x^2$; $-15a^2xz \times -a^3x^3z^2$.

17. $-3mn \times 4m^2n$; $5m^2n^3p \times -mnp^2$; $-28m^3n \times 18mn^2$.

18. $a^2mx \times -am^2x^2$; $m^2nx \times -17m^3n^3$; $-16a^3nx \times -14n^2x^2y^2$.

Case II.—To Multiply Compound Quantities.

31. Rule.—Multiply each term of the multiplicand by each term of the multiplier; the sum of the products thus obtained will be the complete product.

Examples.

$$\begin{array}{l} 5a^2bc - 3ab^2c \\ 4abc \\ \hline 20a^3b^2c^2 - 12a^2b^3c^2 \end{array} \qquad \begin{array}{l} -x^3y + 5x^2y^2 - 7xy^3 \\ -3x^2y \\ \hline 3x^5y^2 - 15x^4y^3 + 21x^3y^4 \end{array}$$

$$\begin{array}{lll} 3x^2 - 2xy + 3y^2 & & \\ 2x^2 + 3xy & & \\ \hline 6x^4 - 4x^3y + 6x^2y^2 & & \\ \quad\quad 9x^3y - 6x^2y^2 + 9xy^3 & & \\ \hline 6x^4 + 5x^3y \quad\quad + 9xy^3 & & \end{array} \qquad \begin{array}{l} a^2 - ax + x^2 \\ ax - x^2 \\ \hline a^3x - a^2x^2 + ax^3 \\ \quad\quad - a^2x^2 + ax^3 - x^4 \\ \hline a^3x - 2a^2x^2 + 2ax^3 - x^4 \end{array}$$

As in addition and subtraction, it is more convenient to work from left to right. We shift the second and following lines to the right, that we may place like terms under one another.

Exercise XIII.

Multiply

1. $3ab - 4xy$ by $2a^2$, $3b^2$, $5ab$, $6x^3$, $4y^2$, $4xy$, $9a^2x^2$, $3b^2y^2$.
2. $-4ay + 5bx$ by $3a$, $4x^2$, $5b$, $7y$, $-2a^2$, $-3b^2x^2$, $-5by^3$.
3. $a^2x - ax^2$ by ax, $-a^2x$, $-ax^2$, a^2x^2, $-a^2x^2$, $4ax^2$, $-3a^2x$.
4. $-a^2 + ab - b^2$ by $3a$, $-2b$, $-a^2$, ab^2, $-ab$, $-a^2b$, $3a^2b^2$.
5. $a^2 - b^2$ by $a + b$, $a - b$, $-a - b$, $a^2 + b^2$, $a^2 - b^2$.
6. $2a - 2b$ by $3a + 3b$, $5a + b$, $a + 2b$, $4a + 5b$.
7. $-4a - 5b$ by $a - 2b$, $3a - b$, $-2a + b$, $-a - b$.
8. $2a^2 - 3x^2$ by $3ax - 3x^2$, $5a^2 + 4ax$, $-a^2 + x^2$.
9. $(x + y)(x + y)$, $(x - y)(x - y)$, $(x + y)(x - y)$.
10. $(x^2 + y^2)(x + y)$, $(x^2 + y^2)(x - y)$, $(x^3 + y^3)(x + y)$.
11. $(x^2 - y^2)(x - y)$, $(x^2 - y^2)(x + y)$, $(x^3 - y^3)(x - y)$.
12. $a^2 + ax + x^2$ by (1.) $a - x$, (2.) $a + x$, (3.) $a^2 - x^2$.
13. $a^2 - ax + x^2$ by (1.) $a + x$, (2.) $a - x$, (3.) $a^2 + x^2$.
14. $a^2 + 2ab + b^2$ by (1.) $a + b$, (2.) $a - b$, (3.) $a^2 - ab$.
15. $a^2 - 2ab + b^2$ by (1.) $a + b$, (2.) $a - b$, (3.) $ab - b^2$.
16. $a^2 - 2ay + y^2$ by (1.) $a^2 - 3ay$, (2.) $2ay - y^2$, (3.) $ay + 2y^2$.
17. $2a^2b^2 - 4abx + 2x^2$ by (1.) $2ab + 2x$, (2.) $-2x - 2ab$.
18. $4a^2x^2 - 8axz + 4z^2$ by (1.) $2ax - 2z$, (2.) $-3ax - 4z$.
19. $3a^2x^2 - 6axy + 3y^2$ by (1.) $5ax + 5y$, (2.) $-4ax - 5y$.
20. $4a^2x^2 - 8axyz + 4y^2z^2$ by (1.) $3ax - 2yz$, (2.) $-2ax + 3yz$.
21. $(x^2 + 4x - 2)(x^2 - 4x - 2)$, $(x^2 - 3x + 2)(x^2 - 3x - 2)$.
22. $(a^2 + ax + x^2)(a^2 - ax - x^2)$, $(a^4 + a^2b^2 + b^4)(a^2 - b^2)$.
23. $(a^2 + 2a - 1)(a^2 - 3a - 1)$, $(a^3 - 3a^2 + 3a - 1)(a^2 - 2a + 1)$.
24. $(3a^2 + ax + x^2)(3a^2 - ax + x^2)$, $(a^2 - 2ax + x^2)(a^2 + 2ax + x^2)$.
25. $(x^5 - 5x + 1)(2x^4 + x - 1)$, $(x^7 - x^4 - x + 1)(x^5 + x^3 - x - 1)$.
26. $(x^3 + 2x^2y + 4xy^2 + 8y^3)(x - 2y)$, $(x^2 + x + 1)(x - 1)$.
27. $(x^3 + 3x^2y + 9xy^2 + 27y^3)(x - 3y)$, $(x^2 - x + 1)(x + 1)$.
28. $(8a^3 + 4a^2x + 2ax^2 + x^3)(2a - x)$.

29. $(a^5 - 2a^4b + 4a^3b^2 - 8a^2b^3 + 16ab^4 - 32b^5)(a + 2b)$.

30. $(a^3 - 2a^2x + 3ax^2 + 4x^3)(a^2 + 2ax - 3x^2)$.

31. $(x^3 + 3x^2y - 2xy^2 + 3y^3)(x^2 + 2xy - 3y^2)$.

32. $(3x^3 + 5x^2 - 7x + 1)(2x^2 - 3x + 1)$.

33. $(a^3 + a^2y + ay^2 + y^3)(a^2 - 2ay + y^2)$.

34. $(a^3 + a^2x + ax^2 + x^3)(a^2 - ax + x^2)$.

35. $(a^2 + b^2 + c^2 - ab - ac - bc)(a + b + c)$.

36. $(a^2 + b^2 + c^2 + ab - ac + bc)(a - b + c)$.

37. $(a + x)(a - x)(a^2 + x^2)$, $(m - 2n)(m + 3n)(n - m)$.

38. $(x^2 + x - 1)(x^2 - x + 1)(x^4 - x^2 + 1)$, $(a - 3x)(3a + x)$ $(2a - 2x)(4x - 3a)$, $(m^3 - m^2n - mn^2)(m^2 + mn - n^2)(m^2 + n^2)$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

32. When the indices of the letters in both multiplicand and multiplier increase or decrease in regular order, it is sufficient to write the coefficients only in multiplying. If a term is wanting, a cipher must be used to stand for it. The first term of the complete product is the product of the first terms of the multiplicand and multiplier; the others follow in regular order.

Examples.

(1.) $(a^4 - 2a^3 + 3a^2 - 2a + 3)(a^2 + 3a + 2)$.

(2.) $(a^5x + 2a^3x^3 - a^2x^4 + 3ax^5)(a^4x^2 + ax^5 - 3x^6)$.

$$\begin{array}{rl}
(1.) & \begin{array}{r}
1-2+3-2+3 \\
1+3+2 \\ \hline
1-2+3-2+3 \\
3-6+9-6+9 \\
2-4+6-4+6 \\ \hline
1+1-1+3+3+5+6
\end{array}
\end{array}
\qquad
\begin{array}{rl}
(2.) & \begin{array}{r}
1+0+2-1+3 \\
1+0+0+1-3 \\ \hline
1+0+2-1+3 \\
1+0+2-1+3 \\
-3-0-6+3-9 \\ \hline
1+0+2+0+0+2-7+6-9
\end{array}
\end{array}$$

The products are thus (1.) $a^6 + a^5 - a^4 + 3a^3 + 3a^2 + 5a + 6$, (2.) $a^9x^3 + 2a^7x^5 + 2a^4x^8 - 7a^3x^9 + 6a^2x^{10} - 9ax^{11}$. In (2.) the term a^4x^2 is wanting in the multiplicand; a^3x^3 and a^2x^4, in the multiplier; and ciphers must be put in their places. In the product, we find ciphers in the places of a^8x^4, a^6x^6, and a^5x^7, which, therefore, do not appear in the answer.

Exercise XIV.

1. Work by this method Exercise XIII. 6, 7, 9, 14, to 20, 22 to 24, and 26 to 34.

2. Exercise XIII. 5, 8, 10 to 13, 21, 25, 37, and 38, observing that some of these have terms wanting either in the multiplicand, or in the multiplier, or in both.

General Results in Multiplication.*

33. The following examples in multiplication should be carefully studied, and the resulting *formulæ* thoroughly committed to memory.

1. (1) $(x+a)(x+b)=x^2+ax+bx+ab=x^2+(a+b)x+ab.$
   (2) $(x-a)(x-b)=x^2-ax-bx+ab=x^2-(a+b)x+ab.$
   (3) $(x+a)(x-b)=x^2+ax-bx-ab=x^2+(a-b)x-ab.$
   (4) $(x-a)(x+b)=x^2-ax+bx-ab=x^2-(a-b)x-ab.$

Hence, if two binomial expressions have a common term, their product *is the square of the common term, the common term multiplied by the sum of the second terms, and the product of the second terms.*

Examples.

$(x+5)(x+3)=x^2+(5+3)x+5.3=x^2+8x+15.$
$(x+5y)(x-3y)=x^2+(5y-3y)x+5y\times(-3y)=x^2+2xy-15y^2.$

2. Substituting a for b in the second expression of 1 (1), (2), (3), we obtain

(1) $(x+a)(x+a)$ or $(x+a)^2=x^2+ax+ax+a^2=x^2+2ax+a^2.$
(2) $(x-a)(x-a)$ or $(x-a)^2=x^2-ax-ax+a^2=x^2-2ax+a^2.$
(3) $(x+a)(x-a) \qquad =x^2+ax-ax-a^2=x^2-a^2.$

Hence we have the following results:

(1) The *square of the sum of two quantities* is equal to *the sum of their squares plus twice their product.*

(2) The *square of the difference of two quantities* is equal to *the sum of their squares* minus *twice their product.*

(3) The *product of the sum and difference of any two quantities* is equal to *the difference of their squares.*

* See page 50.

Examples.

(1) $(3x + y)^2 = (3x)^2 + 2.3x.y + (y)^2$
$= 9x^2 + 6xy + y^2.$

(2) $(2a - 3b)^2 = (2a)^2 - 2.2a.3b + (3b)^2$
$= 4a^2 - 12ab + 9b^2.$

(3) $(4m^2 + 1)(4m^2 - 1) = (4m^2)^2 - 1^2$
$= 16m^4 - 1.$

(4) $(x + y)^2(x - y)^2 = \{(x + y)(x - y)\}^2$
$= (x^2 - y^2)^2 = x^4 - 2x^2y^2 + y^4.$

3. By continued multiplication we obtain the following:
$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$
Hence, if three binomial expressions have a common term, their continued product is *the cube of the common term, the square of the common term multiplied by the sum of the second terms, the common term multiplied by the sum of the products of every pair of the second terms, and the product of the second terms.*

Example.

$(x + 2)(x + 3)(x - 4)$
$= x^3 + (2 + 3 - 4)x^2 + \{2.3 + 2.(-4) + 3.(-4)\}x + \{2.3.(-4)\}$
$= x^3 + x^2 - 14x - 24.$

4. Substituting a for b and c in the continued product $(x + a)(x + b)(x + c)$, we obtain
$(x + a)(x + a)(x + a) = x^3 + (a + a + a)x^2 + (a^2 + a^2 + a^2)x + a^3,$
or $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3.$
Again, by putting $-a$ for a in the above, we obtain
$(x - a)^3 = x^3 + 3(-a)x^2 + 3(-a)^2x + (-a)^3$
$= x^3 - 3ax^2 + 3a^2x - a^3.$

Examples.

(1) $(2x + 3y)^3 = (2x)^3 + 3(2x)^2.3y + 3(2x)(3y)^2 + (3y)^3$
$= 8x^3 + 36x^2y + 54xy^2 + 27y^3.$

(2) $(3a - 4)^3 = (3a)^3 + 3(3a)^2(-4) + 3.3a.(-4)^2 + (-4)^3$
$= 27a^3 - 108a^2 + 144a - 64.$

5. The following examples show that these formulæ may, by bracketing, be applied to other quantities than binomials. Thus,

(1) $(a+b+c)^2 = \{(a+b)+c\}^2 = (a+b)^2 + 2(a+b)c + c^2$
$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2,$
or $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

(2) $(a-b-c)^2 = \{(a-b)-c\}^2 = (a-b)^2 - 2(a-b)c + c^2$
$= a^2 - 2ab + b^2 - 2ac + 2bc + c^2,$
or $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc.$

(3) $(a+b+c)(a+b-c) = (\overline{a+b}+c)(\overline{a+b}-c)$
$= (a+b)^2 - c^2 = a^2 + 2ab + b^2 - c^2.$

(4) $(a^2+ab+b^2)(a^2-ab+b^2) = (\overline{a^2+b^2}+ab)(\overline{a^2+b^2}-ab)$
$= (a^2+b^2)^2 - (ab)^2$
$= a^4 + 2a^2b^2 + b^4 - a^2b^2$
$= a^4 + a^2b^2 + b^4.$

(5) $(a+b+c-d)(a-b+c+d) = (\overline{a+c}+\overline{b-d})(\overline{a+c}-\overline{b-d})$
$= (a+c)^2 - (b-d)^2$
$= a^2 + 2ac + c^2 - b^2 + 2bd - d^2.$

Exercise XV.

1. $(x+7)(x+4)$, $(x-7)(x-4)$, $(x+7)(x-4)$.
2. $(y+5)(y+5)$, $(y-5)(y-5)$, $(y+5)(y-5)$.
3. $(a+5b)(a+3b)$, $(a-5b)(a+3b)$, $(a+b)(a-9b)$.
4. $(x^2+2)(x^2-3)$, $(a^2-5)(a^2-3)$, $(m^2+7)(m^2+1)$.
5. $(ax+4)(ax-3)$, $(mn-5)(mn-9)$, $(ab+cd)(ab-6cd)$.
6. $(a+b)(a-c)$, $(1+a)(1-b)$, $(p^2+q)(p^2+r)$.

7. $(a+x)^2$, $(a-x)^2$, $(x-y)^2$, $(1+x)^2$, $(x-1)^2$.
8. $(x+4)^2$, $(x-3)^2$, $(6+x)^2$, $(2-x)^2$, $(a^2-x^2)^2$.
9. $(x^2+1)^2$, $(x^2-1)^2$, $(2x-4)^2$, $(5+3y)^2$, $(3x-2y)^2$.
10. $(2a+5b)^2$, $(3m-4n)^2$, $(4p+q)^2$, $(y-10z)^2$, $(1-12ab)^2$.
11. $(ab-3cd)^2$, $(1+5a^2b^2)^2$, $(2xy-5z)^2$, $(7m^2n-2)^2$, $(2pq+5r)^2$.
12. $(a^2+3)^2$, $(2x^4-3y^2)^2$, $(5-x^3)^2$, $(y^4-3y^2)^2$, $(a^3-b^2c)^2$.

13. $(a+x)(a-x)$, $(a+1)(a-1)$, $(3a+b)(3a-b)$.
14. $(a^2+1)(a^2-1)$, $(x^2+5)(x^2-5)$, $(a^3-x^4)(a^3+x^4)$.
15. $(a+b)(a-b)(a^2+b^2)$, $(x-1)(x+1)(x^2+1)(x^4+1)$.
16. $(r+2)(r-2)(r^2+4)$, $(y^2-3)(y^2+3)(y^4+9)$.
17. $(m+n)^2(m-n)^2$, $(1-x)^2(1+x)^2$, $(2m-3n)^2(2m+3n)^2$.
18. $\{(x+3y)(x-3y)\}^2$, $\{(3-2pq)(3+2pq)\}^2$.

19. $(a+1)(a+3)(a+5)$, $(m-1)(m-2)(m-3)$.
20. $(x+2)(x+4)(x-6)$, $(xy-5)(xy+1)(xy-3)$.
21. $(r-3)(r-4)(r+5)$, $(p^2-3)(p^2-4)(p^2-5)$.
22. $(a+b)(a+2b)(a+3b)$, $(x-6y)(x+6y)(x-3y)$.
23. $(m-n)(m-4n)(m-7n)$, $(1-x)(1+3x)(1-5x)$.
24. $(x+p)(x+q)(x+r)$, $(a^2-2b^2)(a^2+5b^2)(a^2+8b^2)$.

25. $(x+y)^3$, $(a-b)^3$, $(m+1)^3$, $(p-1)^3$, $(r+2)^3$, $(z-3)^3$.
26. $(2a-b)^3$, $(3x+2y)^3$, $(m-4n)^3$, $(3r-4s)^3$, $(1-5z)^3$.
27. $(ab-3cd)^3$, $(2xy-3z)^3$, $(3-4mn)^3$, $(7pq-1)^3$, $(2a^2b^2-5c)^3$.

28. $(a+b+c)^2$, $(a-b+c)^2$, $(a+b-c)^2$, $(a-b-c)^2$.
29. $(x+2y+3z)^2$, $(x-2y+z)^2$, $(2a+b-5c)^2$, $(m-3n+5p)^2$.
30. $(a+b+c+d)^2$, $(a-b+c-d)^2$, $(a+2b-3c-4d)^2$.
31. $(a^2+b^2-c^2)^2$, $(3x^2-2y^2-z^2)^2$, $(ax+2by-3cz)^2$.
32. $(x^2+x+1)^2$, $(a^2+2a+1)^2$, $(m^2-3m-2)^2$, $(r^2-r+3)^2$.
33. $(3x^2-4x-5)^2$, $b^2+4b-7)^2$, $(2n^2-5n+8)^2$, $(3y^2-6y-9)^2$.
34. $(a^3-a^2+a-1)^2$, $(1-2x+3x^2-4x^3)^2$, $(z^6-2z^4-3z^2+1)^2$.
35. $(a^2+2ab+4b^2)^2$, $(2x^2+5xy+8y^2)^2$, $(m^3+3m^2n+3mn^2+n^3)^2$.

36. $(a+b+c)(a+b-c)$, $(a+b+c)(a-b+c)$.
37. $(a-b+c)(a-b-c)$, $(a+b-c)(a-b-c)$.
38. $(a^2+a+1)(a^2+a-1)$, $(x^2+x+1)(x^2-x+1)$.
39. $(m^2-m+1)(m^2-m-1)$, $(p^2+p+1)(p^2-p-1)$.
40. $(x^2+xy+y^2)(x^2-xy+y^2)$, $(a^4+a^2b^2+b^4)(a^4-a^2b^2+b^4)$.
41. $(a^2+2ab+b^2)(a^2-2ab+b^2)$, $(3a-2b+c)(3a+2b-c)$.
42. $(a+b+c-d)(a+b-c+d)$, $(a-b+c-d)(a-b-c+d)$, $(a+b+c-d)(a-b+c+d)$.
43. $(x^3+x^2y+xy^2+y^3)(x^3+x^2y-xy^2-y^3)$, $(2a+x+3y-z)(2a+x-3y+z)$, $(2a-x+3y-z)(2a-x-3y+z)$.

44. $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.
45. $(p^2+pq+q^2)(p^2-pq+q^2)(p^4-p^2q^2+q^4)$.
46. $(a+b)(a+c)(a+d)(a-b)(a-c)(a-d)$.
47. $(a+b+c)^2-2(a+b+c)(a+b-c)+(a+b-c)^2$.
48. $(x+y-z)^2+2(x+y-z)(x+y+z)+(x+y+z)^2$.
49. $(a+b)^3+3(a+b)^2(a-b)+3(a+b)(a-b)^2+(a-b)^3$.
50. $(x+y)^3-3(x+y)^2(x-y)+3(x+y)(x-y)^2-(x-y)^3$.

DIVISION.

34. The quantity *to be divided* is called the *dividend;* the quantity *we divide by*, the *divisor;* the result, the *quotient* (L. *dividendum*, to be divided; *quotiens*, how often).

Multiplication and division are alike in having two factors and a product, the dividend being the product of the divisor and quotient; for if $12 = 3 \times 4$, then, of course, $12 \div 3 = 4$, and $12 \div 4 = 3$. But multiplication gives the factors to find the product; division, the product and one factor to find the other factor.

35. Rule of Signs.—*Like signs* give *plus*, *unlike signs* give *minus;* as in multiplication.

(1.) $ab \div a = b$, $\because ab = a \times b$; and $-ab \div -a = b$, $\because -ab = -a \times b$.

(2.) $ab \div -a = -b$, $\because ab = -a \times -b$; and $-ab \div b = -a$, $\because -ab = b \times -a$.

From (1), we see that, where *dividend* and *divisor* have the *same sign*, the *quotient* is *positive;* from (2), that, where they have *different signs*, the *quotient* is *negative.*

Similarly, $12ab \div 3a = 4b$, and $12ab \div 4b = 3a$, $\because 12ab = 3a \times 4b$; $-6abcd \div -2ac = 3bd$, and $-6abcd \div 3bd = -2ac$, $\because -6abcd = -2ac \times 3bd$; $20abxy \div -4ax = -5by$, and $20abxy \div -5by = -4ax$, $\because 20abxy = -4ax \times -5by$.

36. Rule of Indices.—To divide one power of a quantity by another, subtract the index of the divisor from that of the dividend.

$$a^2 \div a = a^{2-1} = a, \ \because a^2 \div a = a \, . \, a \div a = a.$$

$$a^4 \div a^2 = a^{4-2} = a^2, \ \because a^4 \div a^2 = aa \, . \, aa \div aa = a^2.$$

$$a^5 \div a^2 = a^{5-2} = a^3, \ \because a^5 \div a^2 = aaa \, . \, aa \div aa = a^3.$$

Similarly, $21a^3b^2 \div 7ab = 3a^2b$, $21a^3b^2 \div 3a^2b = 7ab$;

$$-20a^3b^7 \div 5a^2b^3 = -4ab^4, \ -20a^3b^7 \div -4ab^4 = 5a^2b^3.$$

CASE I.—TO DIVIDE ONE SIMPLE QUANTITY BY ANOTHER.

37. RULE.—Write the dividend as the numerator, and the divisor as the denominator, of a fraction; cast out of both of them the factors common to both, observing the rule of signs and the rule of indices: the factors left will be the quotient.

Examples.

$\dfrac{12a^3b^4}{-4a^2b^2} = -3ab^2$, casting out of both the factors, -4, a^2, and b^2.

$\dfrac{10a^3x^2z}{-14a^2x^3z^4} = \dfrac{-5a}{7xz^3}$, casting out the factors, -2, a^2, x^2, and z.

This process is the same as cancelling in arithmetic. When the dividend contains all the factors of the divisor, the quotient can be discovered at once by inspection.

EXERCISE XVI.

1. $ab \div a$, $abc \div b$, $abcd \div ac$, $abxy \div by$, $ax \div ax$.

2. $6ax \div 3x$, $9ax \div 3a$, $8xy \div 8xy$, $8xy \div 4xy$, $14abxz \div 7bz$.

3. $-12xyz \div 4xy$, $-30axy \div 5ay$, $-21abxyz \div 3by$, $-8abc \div 8ac$.

4. $15abc \div 5ab$, $16acd \div -8ad$, $33abcxy \div -3by$, $11axz \div -az$.

5. $-12axz \div 6xz$, $18xyz \div -9xz$, $15cde \div -cde$, $-27abxy \div 9ax$.

6. $-8abcd \div -4ac$, $-5axyz \div -axyz$, $-7cdxz \div -7cx$.

7. $-3xyz \div -3x$, $-15abx \div -3bx$, $-15abx \div -abx$.

8. $24abcy \div -8ay$, $25cdx \div 5cx$, $-27abc \div 9ac$, $-33abcd \div -11acd$.

9. $abcz \div -bz$, $-acxz \div ax$, $5ay \div 5ay$, $7cd \div -7cd$.

10. $-3axy \div 3ax$, $-6axy \div -6axy$, $-abcd \div abcd$.

11. $a^7 \div a^4$, $a^6 \div a^2$, $a^3 \div a$, $a^2 \div a$, $a^3 \div a^2$, $a^5 \div a^4$, $a^9 \div a^7$.

12. $a^3b^5 \div ab^3$, $a^4b^2 \div -ab^2$, $a^3b^3 \div a^3b$, $-a^7b^8 \div -a^5b^3$, $a^5b^3 \div ab^2$.

13. $a^4x^5y^3 \div a^2x^2y^2$, $-a^4b^4c^4 \div a^2b^3c$, $-a^2b^2c^2 \div -a^2bc$.

14. $a^2x^2z^3 \div -axz^2$, $abc^4d \div -ac^2$, $-abx^2y^3 \div -bx^2y$.

15. $8a^5 \div 4a^2$, $-6a^4 \div 3a^3$, $10a^2 \div -5a$, $-12a^2x^2 \div 4ax$.

16. $10a^2x^3 \div -2ax^2$, $8a^3x^2 \div -8a^2x$, $9a^3x^3 \div a^2x^2$, $-5a^4x^3 \div a^3x^3$.

17. $-a^4x^3 \div -a^2x$, $7a^3x^4 \div -a^3x^4$, $8a^3x^5 \div -8a^3x^3$.

18. $2m^3 \div -2m$, $-5m^4n^3 \div -5m^3n^3$, $7m^2n^3 \div -7m^2n^3$, $-3np^3 \div np^2$.

19. $abc \div acd$, $-axy \div axz$, $abx \div -acx$, $10ax \div 5bx$, $3ab \div 4ac$.

20. $a^2 \div a^5$, $a^3 \div -a^4$, $-a^3 \div -a^7$, $5a^2 \div 5a^4$, $-4a^3 \div -7a^5$.

21. $ax \div axy$, $a^3x^2 \div a^2x^3$, $6abx \div 7acx$, $-4a^2x^3 \div 5a^3x^3$.

22. $a^2x^3 \div -a^3x^4$, $-ax^2 \div -a^2x$, $4a^3x \div -8a^4x^3$.

Case II.—To Divide a Compound Quantity by a Simple One.

38. Rule.—Divide each term of the dividend by the divisor.

Examples.

$$\frac{2a^4 - 4a^3 + 6a^2}{2a^2} = a^2 - 2a + 3,$$

$$\frac{6a^2 - 5ax + x^2}{-3ax} = \frac{-2a}{x} + \frac{5}{3} - \frac{x}{3a}.$$

Exercise XVII.

Divide

1. $ab + ac$ by a, $ax + xy$ by x, $ay + xy$ by y, $ab - bc$ by b.
2. $abc + bcd$ by bc, $abc - acd$ by ac, $axy - axz$ by ax.
3. $-ab + ax$ by a, $-cd + de$ by d, $-xy + yz$ by y.
4. $-ax - xy$ by x, $-ac - bc$ by c, $-ay - yz$ by y.
5. $-ab + ax$ by $-a$, $-ab - ac$ by $-a$, $xy - yz$ by $-y$.
6. $12ab + 4ac$ by $2a$, $-18xz + 9yz$ by $-3z$.
7. $20axy - 15xyz$ by $-5xy$, $-33abcxy + 22bcxyz$ by $-11bcxy$.
8. $18abc - 24acd$ by $-6ac$, $-axz + 8ayz$ by az.
9. $-18abc - 9ab$ by $-9ab$, $3abc - 3acd$ by $-3ac$.
10. $-5abxy + 10axyz$ by $-5axy$, $7abcxy - 7abcx$ by $-7abcx$.
11. $a^3 + a^4$ by a^2, $a^4 - a^3$ by a^2, $a^3 + a^2$ by a^2, $a^5 - a^3$ by a^3.
12. $a^3b^3 + a^3b^5$ by a^2b^2, $a^3b^4 - a^4b^3$ by a^2b^2, $-a^4b^5 + a^5b^4$ by a^3b^3.
13. $a^2b^2x^2 + a^3b^3x^3$ by abx, $a^5b^2x^3 - a^5x^3y^2$ by $-a^3x^2$.

14. $-a^3x^3 + a^2x$ by $-a^2x$, $-a^2b^3x^3 - a^2b^2x$ by $-a^2b^2x$.

15. $6a^5x - 9a^6x$ by $3a^2x$, $-4a^4x^6 + 8a^5x^7$ by $4a^2x^4$.

16. $-6a^3b^4 + 6a^4b^5$ by $-3a^2b^3$, $5a^4x^5 - 5a^2x^3$ by $5a^2x^3$.

17. $x^3 - 2x^2 + x$ by x, $x^5 - x^4 + x^3 - x^2$ by x^2, $a^2x^3 + b^2x^4 - c^2x^5$ by x^2.

18. $3m^3n^2p - 15m^3n^7$ by $3m^2n^2$, $16m^3nx - 8m^2n^2 + 4m^2n^3$ by $4m^2n$.

19. $abx - acx$ by adx, $axyz + abcz$ by abx, $8axy - 3byz$ by $4ay$.

20. $m^3 - m^2n + mn^2 - n^3$ by m^2n^2, $m^4 + 6m^2n^2 - 3mn^3 + n^4$ by $3m^2n^2$.

Case III.—To Divide One Compound Quantity by Another.

39. Rule.—(1.) Arrange the terms of both dividend and divisor according to the powers of some letter in both; (2.) find how often the first term of the divisor is contained in the first term of the dividend, and set down the result as the first term of the quotient; (3.) multiply each term of the divisor by this term of the quotient, and subtract the result from the dividend; (4.) consider the remainder a new dividend, and go on as before.

The divisor, dividend, and every new remainder, must all, to prevent confusion, be arranged according to powers of the *same letter*, and they must all be arranged in *the same way*, either *all* by *ascending* powers, or *all* by *descending* powers, of the letter chosen (called *the letter of reference*). And the division must be continued till there is no remainder, or till the highest power of this letter in the remainder be lower than its highest power in the divisor. The other letters should be kept in alphabetical order.

The remaining terms of the dividend, after the first subtraction, are brought down just as they are needed. In example (3), $-25a^2x^4$ is brought down after the first subtraction; $+12ax^5$, after the second.

If divisor and dividend be *both* multiplied or *both* divided by the *same* quantity, the quotient is not altered. Therefore, where the divisor begins with a minus quantity, first multiply both by -1, that is, change *all* their *signs*. In XVIII., 9 (1), first divide both by 4; in 9 (2), by 2; &c.

Example (2) is exactly the same as (1), and (4) exactly the same as (3); but in (1) and (3) all the parts are arranged according to *descending* powers of a; in (2) and (4), according to *ascending* powers of a. In (3) and (4), to save space, the quotient is placed under the divisor.

Examples.

(1.)

$$a^2 - 3a + 1)\,a^4 - 4a^3 + 4a^2 - a\,(a^2 - a$$
$$a^4 - 3a^3 + a^2$$
$$- a^3 + 3a^2 - a$$
$$- a^3 + 3a^2 - a$$

(2.)

$$1 - 3a + a^2)\,- a + 4a^2 - 4a^3 + a^4\,(- a + a^2$$
$$- a + 3a^2 - a^3$$
$$a^2 - 3a^3 + a^4$$
$$a^2 - 3a^3 + a^4$$

(3.)

$$\begin{array}{r|l} 3a^2 + 4ax - 3x^2 & -6a^5x + a^4x^2 + 6a^3x^3 - 25a^2x^4 + 12ax^5 \\ -2a^3x + 3a^2x^2 - 4ax^3 & -6a^5x - 8a^4x^2 + 6a^3x^3 \end{array}$$
$$9a^4x^2 - 25a^2x^4$$
$$9a^4x^2 + 12a^3x^3 - 9a^2x^4$$
$$- 12a^3x^3 - 16a^2x^4 + 12ax^5$$
$$- 12a^3x^3 - 16a^2x^4 + 12ax^5$$

(4.)

$$\begin{array}{r|l} -3x^2 + 4ax + 3a^2 & 12ax^5 - 25a^2x^4 + 6a^3x^3 + a^4x^2 - 6a^5x \\ -4ax^3 + 3a^2x^2 - 2a^3x & 12ax^5 - 16a^2x^4 - 12a^3x^3 \end{array}$$
$$- 9a^2x^4 + 18a^3x^3 + a^4x^2$$
$$- 9a^2x^4 + 12a^3x^3 + 9a^4x^2$$
$$6a^3x^3 - 8a^4x^2 - 6a^5x$$
$$6a^3x^3 - 8a^4x^2 - 6a^5x$$

Exercise XVIII.

Divide

1. $x^2 + 15x + 56$ by $x + 8$, $x^2 + 9x + 8$ by $x + 1$.
2. $x^2 - 10x + 24$ by $x - 6$, $x^2 - 9x + 8$ by $x - 8$.
3. $x^4 - x^2y - 12y^2$ by $x^2 - 4y$, $x^8 + x^4 - 2$ by $x^4 - 1$.
4. $x^4 - 2x^2y - 3y^2$ by $x^2 + y$, $x^6 - x^3y - 20y^2$ by $x^3 + 4y$.
5. $a^4 + a^3b - ab^3 - b^4$ by (1.) $a + b$, (2.) $a - b$, (3.) $a^2 - b^2$.
6. $a^4 + a^3b - 2a^2b^2 - ab^3 + b^4$ by (1.) $a + b$, (2.) $a - b$, (3.) $a^2 - b^2$.

7. $-a^4+a^3b-ab^3+b^4$ by (1.) $-a+b$, (2.) $a+b$, (3.) $-a^2+b^2$.

8. $12a^2+23ax+10x^2$ by $3a+2x$, $16a^4-81x^4$ by $4a^2-9x^2$.

9. $12a^2b^2-48x^2y^2$ by $4ab+8xy$, $6x^3-6xy^2-36y^3$ by $2x-4y$.

10. $6a^3-24a^2x+48ax^2-48x^3$ by $3a-6x$, $3x^3-3xy^2-18y^3$ by $x-2y$, $x^6-3x^4+3x^2-1$ by x^2+1.

11. a^4-x^4 by $a+x$, $a^{12}-x^{12}$ by a^3+x^3, $16a^4-81x^4$ by $2a-3x$.

12. $12x^4-8x^3y-8x^2y^2+20xy^3-16y^4$ by $3x+4y$.

13. $6x^4-12x^2y^2+18xy^3-12y^4$ by $2x+4y$, $a^{10}+x^{15}$ by a^2+x^3.

14. $x^{10}-5x^8+10x^6-10x^4+5x^2-1$ by x^2-1, x^5+32 by $x+2$.

15. $x^{12}-6x^{10}+15x^8-20x^6+15x^4-6x^2+1$ by x^2-1.

16. a^5+x^5 by $a+x$, a^6-x^6 by $a-x$, x^6-64 by $x-2$.

17. $x^6-3x^4+3x^2-1$ by x^4-2x^2+1, x^4-x^2+2x-1 by x^2+x-1.

18. $64a^4+12a^2b^2+b^4$ by $8a^2+2ab+b^2$, x^4+3x^2+1 by x^2-x-1.

19. $144a^4-36a^2b^2+36ab^3-9b^4$ by $12a^2+6ab-3b^2$.

20. $2x^4-16x^3+46x^2-56x+24$ by x^2-4x+3, x^6-64 by x^3+8.

21. $3a^4-12a^3-24a^2+72a-480$ by $a^2-2a-20$.

22. $a^2-b^2+2bc-c^2$ by $a+b-c$, $4a^2-8ab+4b^2-4c^2$ by $a-b-c$.

23. $\dfrac{4a^6x^2-16a^5x^3+16a^4x^4-4a^2x^6}{2a^3x-4a^2x^2-2ax^3}$, $\dfrac{12a^4-72a^2x^2+12x^4}{4a^2+8ax-4x^2}$.

24. $\dfrac{4a^4-14a^3x+20a^2x^2-14ax^3+4x^4}{2a^2-3ax+2x^2}$, $\dfrac{6a^6b^2-12a^4b^4+6a^2b^6}{3a^3b+6a^2b^2+3ab^3}$.

25. $\dfrac{a^2x^6-4a^3x^5+6a^4x^4-4a^5x^3+a^6x^2}{ax^3-2a^2x^2+a^3x}$, $\dfrac{x^5-5x^3+7x^2+8x+1}{x^2+3x+1}$.

26. $\dfrac{x^5-3x^3+5x^2-4x+1}{x^2+2x-1}$, $\dfrac{a^5-7a^3+3a^2-1}{a^2-2a-1}$, $\dfrac{a^5-5a^3+5a^2-1}{a^2+3a+1}$.

27. $\dfrac{6a^5-22a^3b^2+44a^2b^3-96ab^4+64b^5}{3a^2+6ab-8b^2}$.

28. $\dfrac{-6a^5+6a^3x^2-6a^2x^3+6x^5}{-2a^2-4ax-2x^2}$, $\dfrac{8a^5+8a^3b^2-64a^2b^3-64b^5}{4a^2+8ab+16b^2}$.

29. $\dfrac{x^{10}-5x^8+10x^6-10x^4+5x^2-1}{x^4-2x^2+1}$, $\dfrac{4a^5-8a^2x^3-16x^5}{2a^2+4ax+4x^2}$.

30. $\dfrac{x^{12} - 6x^{10} + 15x^8 - 20x^6 + 15x^4 - 6x^2 + 1}{x^4 - 2x^2 + 1}$, $\dfrac{16a^4 - 81x^4}{2a + 3x}$.

31. $\dfrac{a^6 - 6a^5x + 15a^4x^2 - 16a^3x^3 + 7a^2x^4 - 2ax^5 + x^6}{a^2 - 2ax + x^2}$, $\dfrac{a^{16} - x^{16}}{a^2 - x^2}$.

32. $\dfrac{x^6 - x^4 + 5x^3 - x + 2}{x^3 + x^2 - x + 1}$, $\dfrac{x^5 - 5x^3 + 5x^2 - 1}{x^3 - 3x^2 + 3x - 1}$, $\dfrac{x^5 - 7x^3 + 3x^2 - 1}{x^3 + 2x^2 - 2x + 1}$.

33. $\dfrac{x^6 - 4x^5 + 4x^4 - 4x^2 + 4xy - y^2}{x^3 - 2x^2 + 2x - y}$, $\dfrac{6m^6 + 48m^2n^4 - 6n^6}{2m^3 + 4m^2n + 4mn^2 - 2n^3}$.

34. $\dfrac{a^8 + a^6b^2 - a^2b^6 - b^8}{a^3 - a^2b + ab^2 - b^3}$, $\dfrac{a^2 - b^2 - c^2 + d^2 + 2ad + 2bc}{a + b - c + d}$.

35. $\dfrac{a^2 - 2ax + x^2 - y^2 + 2yz - z^2}{a - x + y - z}$, $\dfrac{-a^3 + b^3 + c^3 + 3abc}{a - b - c}$.

36. $\dfrac{a^3 + b^3 - c^3 + 3abc}{a + b - c}$, $\dfrac{a^3 - b^3 - c^3 - 3abc}{a - b - c}$, $\dfrac{x^3 - y^3 + z^3 + 3xyz}{x - y + z}$.

Division by Detached Coefficients.

40. This method can be used in division under the same conditions as in multiplication. See section 32.

Examples.

(1.) $(8a^4 - 2a^2x^2 + 8x^4) \div (2a^2 - 3ax + 2x^2) = 4a^2 + 6ax + 4x^2$.

(2.) $(4x^6 + 6x^4 - 2x^3 - 3x) \div (2x^4 - x) = 2x^2 + 3$.

```
         (1.)                                  (2.)
2-3+2)8+ 0- 2+ 0+8             2+0+0-1)4+0+6-2+0-3
4+6+4 8-12+ 8                  2+0+3    4+0+0-2
      ---------                         -------
        12-10+ 0                          0+6+0+0
        12-18+12                          0+0+0+0
        --------                          -------
            8-12+8                          6+0+0-3
            8-12+8                          6+0+0-3
```

Exercise XIX.

Work, by this method, Exercise XVIII., except 22, 33–36. Sums 1–4, 12, 15, 21; the first in 8, 10, 14, 17, 20, 24, 25, 29, 31; and some others, have no terms wanting.

General Results in Division.*

41. The following results enable us to write down from inspection the quotients of similar sums in division, and should be carefully committed to memory. The index n stands for any whole number; x^n means the n^{th} power of x, and is read x to the n^{th}; by 'divisible' we mean divisible without remainder.

1. (1.) When two quantities are raised to the *same odd power*, the *difference of these powers* is divisible by *the difference of the quantities themselves*, and the *sum of the powers* by the *sum of the quantities themselves*; (2.) when they are raised to the *same even power*, the *difference of the powers* is divisible by *either the sum* or *the difference of the quantities themselves*, but *the sum of the powers* is divisible by *neither*. This is expressed in *Formulæ* thus,

1.	$x^n - y^n$	is	divisible by	$x - y$	when	n is	an	*odd*	number.
	$x^n + y^n$	"	"	$x + y$	"	"	"	"	"
2.	$x^n - y^n$	"	"	either	"	"	*even*	"	
	$x^n + y^n$	"	"	neither	"	"	"	"	

From this it follows that

x^3-y^3, x^5-y^5, x^7-x^7, x^9-y^9, &c.	are divisible	by	$x-y$,
x^3+y^3, x^5+y^5, x^7+y^7, x^9+y^9, &c.	"	"	$x+y$,
x^2-y^2, x^4-y^4, x^6-y^6, x^8-y^8, &c.	"	"	either.
x^2+y^2, x^4+y^4, x^6+y^6, x^8+y^8, &c.	"	"	neither.

It matters not what the quantities themselves are, provided they are *both* raised to *the same power*. Thus, since x^4 and 1 are the 2d powers of x^2 and 1; a^8 and b^8, the 4th powers of a^2 and b^2; $8a^6$ and $27b^3$, the 3d powers of $2a^2$ and $3b$; a^{15} and $32b^{10}$, the 5th powers of a^3 and $2b^2$; therefore

x^4-1 is divisible by	x^2+1 or x^2-1;	a^8-b^8 by	a^2+b^2 or a^2-b^2;
$8a^6+27b^3$ "	$2a^2+3b$;	$a^{15}+32b^{10}$ by	a^3+2b^2;
$8a^6-27b^3$ "	$2a^2-3b$;	$a^{15}-32b^{10}$ by	a^3-2b^2.

2. The quotient consists of terms *regularly decreasing* in powers of the *first* quantity, and *regularly increasing* in powers of the *second* quantity; where the divisor is the *difference* of the quantities, the terms of the quotient are all $+$; where it is their *sum*, they are alternately $+$ and $-$.

* See page 50.

Examples.

$$\frac{x^3-y^3}{x-y}=x^2+xy+y^2; \qquad \frac{x^5+1}{x+1}=x^4-x^3+x^2-x+1.$$

$$\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3; \quad \frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3.$$

$$(8a^6-27b^3)\div(2a^2-3b)=4a^4+6a^2b+9b^2.$$

$$(a^{15}+32b^{10})\div(a^3+2b^2)=a^{12}-2a^9b^2+4a^6b^4-8a^3b^6+16b^8.$$

Exercise XX.

Tell by inspection the quotients of

1. a^3+b^3, a^5+b^5, a^2-b^2, a^4-b^4, a^6-b^6, each by $a+b$.
2. a^3-b^3, a^5-b^5, a^2-b^2, a^4-b^4, a^6-b^6, each by $a-b$.
3. x^3+1, x^5+1, x^2-1, x^4-1, x^6-1, each by $x+1$.
4. x^3-1, x^5-1, x^2-1, x^4-1, x^6-1, each by $x-1$.
5. x^6-y^6 by $x+y$, $x-y$, x^2-y^2, x^3+y^3, x^3-y^3.
6. x^8-y^8 by $x+y$, $x-y$, x^2+y^2, x^2-y^2, x^4+y^4, x^4-y^4.
7. $x^{10}-y^8$ by x^5+y^4, and x^5-y^4; x^9-y^6 by x^3-y^2.
8. $4m^2-9n^2$ by $2m+3n$; $64x^3+1$ by $4x+1$.
9. $(a+x)^2-y^2$ by $a+x-y$, $a^2-(x-y)^2$ by $a-x+y$.
10. By what quantities are (1) a^2-x^2, (2) a^2+x^2, (3) a^3-x^3, (4) a^3+x^3, (5) a^4+x^4, (6) a^4-x^4, (7) a^5+x^5, (8) a^5-x^5, (9) a^6-x^6, (10) $a^{12}-x^9$, (11) x^2-1, (12) x^3+1, (13) x^5-1, (14) x^8-1, divisible.

Resolution into Factors.

42. It is often necessary to resolve algebraical expressions into their elementary factors (section 7), and this is most frequently done by using the general results given in sections 33 and 41. The principal cases of resolution are:

1. Where all the terms have a common factor. Thus,

(1) $abx^3-acx^2=ax^2(b-c)$.

(2) $-2x^3-4x^2+8x=-2x(x^2+2x-4)$.

In resolving, expressions should first be cleared of these factors.

2. Where the expression consists of two sets of terms, which, on being resolved, are found to have a common factor. Thus,

(1) $ab + 3a + 2b + 6 = a(b + 3) + 2(b + 3) = (a + 2)(b + 3)$.
(2) $mx + my - nx - ny = m(x + y) - n(x + y) = (m - n)(x + y)$.

3. Where the expression is a trinomial, having the square of some quantity in the first term, and the quantity itself in the second, but wanting it in the third. Such a trinomial (called a *quadratic trinomial*) can be resolved when the coefficient of its second term is the sum of two quantities, and its third term their product.

Thus, by Formulæ 1 (1–4), sect. 33,

(1) $x^2 + 5x + 6 = x^2 + (3 + 2)x + 3.2 = (x + 3)(x + 2)$.
(2) $x^2 - 7x + 12 = x^2 + (-4 - 3)x + (-4)(-3) = (x - 4)(x - 3)$.
(3) $x^2 - 2x - 3 = x^2 + (-3 + 1)x + (-3).1 = (x - 3)(x + 1)$.
(4) $x^2 + 2xy - 8y^2 = x^2 + (4y - 2y)x + 4y(-2y) = (x + 4y)(x - 2y)$.

For the second terms of the factors we require to find two quantities, having

in (1)	6 for their product and	5	for their sum,	∴ 3 and 2.
(2)	12 „ „ „	-7	„	∴ -4 and -3.
(3)	-3 „ „ „	-2	„	∴ -3 and 1.
(4)	$-8y^2$ „ „ „	$2y$	„	∴ $4y$ and $-2y$.

When the last term of the trinomial is positive, both quantities, as in (1) and (2), have the sign of its second term; when negative, the quantities, as in (3) and (4), differ in sign, the greater having that of the second term.

4. Where the expression is, or contains, the square of a quantity. Thus by Formulæ 2 (1), (2), sect. 33,

(1) $a^2 + 2ab + b^2 = (a + b)(a + b)$.
(2) $4x^2 - 12xy + 9y^2 = (2x - 3y)(2x - 3y)$.
(3) $3a^3x - 6a^2bx + 3ab^2x = 3ax(a^2 - 2ab + b^2) = 3ax(a - b)(a - b)$.

5. Where the expression is, or contains, the difference of the squares of two quantities. Thus, by Formula 2 (3), sect. 33,

(1) $16a^2 - 25b^2 = (4a + 5b)(4a - 5b)$.
(2) $a^3 - 9ab^2 = a(a^2 - 9b^2) = a(a + 3b)(a - 3b)$.
(3) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$.

6. Where the expression is the sum or difference of the cubes of two quantities. By sect 41,

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2, \therefore x^3+y^3=(x+y)(x^2-xy+y^2) \quad (1)$$

$$\frac{x^3-y^3}{x-y}=x^2+xy+y^2, \therefore x^3-y^3=(x-y)(x^2+xy+y^2) \quad (2)$$

Thus,

$$(1)\ a^3+8b^3=a^3+(2b)^3=(a+2b)\{a^2-(a\times 2b)+(2b)^2\}$$
$$=(a+2b)(a^2-2ab+4b^2);$$

$$(2)\ 8a^3-27=(2a)^3-3^3=(2a-3)\{(2a)^2+(2a\times 3)+3^2\}$$
$$=(2a-3)(4a^2+6a+9).$$

Expressions like a^6-b^6 may be resolved by applying sect. 33, 2.(3), and sect. 41, as above. Thus,

$$a^6-b^6=(a^3+b^3)(a^3-b^3)=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2).$$

7. Where the expression is compound, but, by the use of brackets, capable of being expressed as the difference of two squares. Resolve as in case 5. See sect. 33, 5.(3), (4), (5). Thus,

$$(1)\ a^2+4ab+4b^2-9c^2=(a+2b)^2-(3c)^2$$
$$=(a+2b+3c)(a+2b-3c).$$

$$(2)\ 1-x^2+2xy-y^2=1-(x^2-2xy+y^2)$$
$$=1^2-(x-y)^2$$
$$=(1+x-y)(1-x+y).$$

$$(3)\ a^2-2ab+b^2-c^2-2cd-d^2=(a^2-2ab+b^2)-(c^2+2cd+d^2)$$
$$=(a-b)^2-(c+d)^2$$
$$=(a-b+c+d)(a-b-c-d).$$

$$(4)\ m^4+m^2n^2+n^4=(m^4+2m^2n^2+n^4)-m^2n^2=(m^2+n^2)^2-(mn)^2$$
$$=(m^2+n^2+mn)(m^2+n^2-mn)$$
$$=(m^2+mn+n^2)(m^2-mn+n^2).$$

Exercise XXI.

Resolve into elementary factors:

1. a^2b+a^2c, $6x^2-9x$, $ax^2+bx^2-cx^2$, $2abx-4acx-6adx$.

2. $m^3+2m^2n+3mn^2$, $2a^2c-4abc+6b^2c$, $a^2bx^3+a^2bxy-a^2by^2$.

3. $-a^2x - x^2y$, $-x^4 + x^3 - x^2 + x$, $x^3y - x^2y^2 + xy^3$.

4. $-10x^3y^3 + 15x^2y^4 - 20xy^5$, $-60a^5b^3 + 72a^4b^4 - 84a^3b^5$.

5. The answers to Exercise IX. 2–9; VIII. 19–23, &c.

6. $ax + bx + ay + by$, $xy + y + 3x + 3$, $ab + 4a + 3b + 12$.

7. $ab - bx + ax - x^2$, $3mx - nx + 9my - 3ny$, $ax - 2a + 3x - 6$.

8. $mx - nx - my + ny$, $ap - aq - bp + bq$, $yz - y - 2z + 2$.

9. $amx + a^2my + nx + any$, $abx - cdx + aby - cdy$.

10. $axy + bcxy - 2az - 2bcz$, $2mx^2 - 2mxy - 2nxy + 2ny^2$.

11. $x^2 + 8x + 15$, $a^2 + 8a + 7$, $y^2 + 11y + 18$, $m^2 + 11m + 24$, $p^2 + 12p + 20$, $a^2b^2 + 10ab + 21$, $x^4 + 11x^2 + 28$, $x^2 + 13xy + 30y^2$.

12. $x^2 - 7x + 10$, $a^2 - 7a + 6$, $y^2 - 12y + 27$, $m^2 - 13m + 36$ $n^2 - 15n + 50$, $x^2y^2 - 8xy + 12$, $a^4 - 14a^2 + 33$, $a^2 - 13ab + 40b^2$.

13. $x^2 + 5x - 14$, $x^2 + 4x - 21$, $b^2 + b - 20$, $n^2 + 5n - 24$, $r^2 + 4r - 21$, $p^2q^2 + 7pq - 30$, $z^4 + 9z^2 - 22$, $m^2 + 11mn - 12n^2$.

14. $x^2 - 5x - 14$, $y^2 - 7y - 8$, $c^2 - c - 30$, $m^2 - 8m - 33$, $z^2 - 2z - 35$, $m^2n^2 - mn - 42$, $n^4 - 3n^2 - 54$, $p^2 - 2pq - 63q^2$.

15. $3x^3 - 15x^2 + 12x$, $4a^2 + 16a + 12$, $2m^2 - 10m + 12$, $3x^2y^2z + 27xyz + 42z$, $a^2b^2 - 3abxy - 70x^2y^2$, $45 - 14ab + a^2b^2$, $36c^2 - 15a^2bc + a^4b^2$.

16. $a^2 + 2ax + x^2$, $x^2 + 6xy + 9y^2$, $x^2 + 10x + 25$, $a^2b^2 + 14ab + 49$.

17. $x^2 - 2xy + y^2$, $m^2 - 4mn + 4n^2$, $p^2 - 16p + 64$, $x^2y^2 - 18xy + 81$.

18. $4m^2 - 12mn + 9n^2$, $x^4 - 2x^2 + 1$, $x^6 + 2x^3y^2 + y^4$.

19. $4m^2n^2 - 20mnp + 25p^2$, $1 - 8a^2b + 16a^4b^2$, $9x^6y^4z^2 - 6x^3y^2z + 1$.

20. $3a^2x - 6abx + 3b^2x$, $x^2y - 2xy + y$, $5m^2 + 20mn + 20n^2$.

21. $a^2 - x^2$, $x^2 - y^2$, $m^2 - n^2$, $x^2 - 4$, $x^2 - 9$, $x^2 - 16$, $25 - x^2$.

22. $x^2 - 1$, $4x^2 - 1$, $1 - 9x^2$, $1 - 25x^2$, $x^2y^2 - y^2$, $a^2 - a^2x^2$.

23. $4a^2 - 4b^2$, $a^2 - 25b^2$, $16a^2 - 36b^2$, $36a^2x^2 - 49b^2y^2$.

24. $a^4 - x^4$, $x^4 - y^4$, $m^4 - n^4$, $x^4 - 16$, $81 - x^4$, $a^8 - x^8$, $x^8 - y^8$, $m^{16} - n^{16}$.

25. $16a^2x^2 - 9x^4$, $8a^3x^2y - 27ax^2y^3$, $a^4 - 9b^2$, $16y^2 - x^6$, $a^4b^6 - x^8$.

26. $a^3 + b^3,\ 1 + x^3,\ x^3 + 8,\ 27 + y^3,\ a^3b^3 + 64,\ 27z^3 + 8.$

27. $a^3 - b^3,\ x^3 - 1,\ 8 - x^3,\ m^3 - 27,\ 64 - x^3y^3,\ 27 - 64n^3.$

28. $a^3b^3 + x^3y^3,\ x^3y^3 - 125z^3,\ 1 - 216a^3b^6,\ m^6n^3 - 343.$

29. $3x^5 + 3x^2y^3,\ a^5x - a^2x^4,\ x^3y^2 - 125y^2,\ 8m^4 - 125mn^3.$

30. $a^6 - x^6,\ x^6 - z^6,\ x^6 - 1,\ 1 - x^6,\ a^6 - 64,\ 729 - x^6,\ x^{12} - y^6.$

31. $(a + b)^2 - c^2,\ x^2 - (y + z)^2,\ m^4 - (m + 1)^2,\ p^2 - (q - r)^2.$

32. $(a + b)^2 - (c + d)^2,\ (x - y)^2 - (z - 1)^2,\ (a + b)^2 - (x - y)^2.$

33. $(x - 2y)^2 - 9z^2,\ (a^2 + b^2)^2 - a^2b^2,\ (3m + 2n)^2 - (3m - 2n)^2.$

34. $a^2 - 2ab + b^2 - c^2,\ 1 + 2x + x^2 - x^4,\ x^2 + 6xy + 9y^2 - 25z^2.$

35. $a^2 - b^2 - 2bc - c^2,\ x^2 - y^2 + 2yz - z^2,\ a^4 - a^2 - 2a - 1.$

36. $9a^2 - 4b^2 - 4bc - c^2,\quad 2ab - a^2 - b^2 + 1,\quad 4yz + x^2 - y^2 - 4z^2.$

37. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2,\ m^2 + n^2 - p^2 - q^2 - 2mn + 2pq.$

38. $a^2 + 4ax + 4x^2 - 9b^2 + 24by - 16y^2,\ a^2 - (2b - 3c + 4d)^2.$

39. $x^4 + x^2y^2 + y^4,\ a^4 + a^2 + 1,\ a^8 + a^4b^4 + b^8.$

40. $(a + b)^2 - 2(a + b)c + c^2,\ x^2 + 2x(y + z) + (y + z)^2.$

41. $5(a^2 - b^2) + 3(a + b)^2;\ (x^3 + y^3) + xy(x + y).$

42. $2(x^2 - y^2) + 3(x^2 - 3xy + 2y^2);\ (m + n)^2 + 3(m^2 + mn).$

43. $(a + b)^2 - 4(a^2 + ab) + 3(a^2 - b^2) + 3(ab + b^2).$

44. $(x^3 - y^3) - x(x^2 - y^2) + y(x - y)^2.$

LITERAL COEFFICIENTS.

43. Revise sects. 7, 19, 20–22, 26, 27, and 42 (1–2).

In (1) the quantities ax, bx, and $-x$ differ only in their signs and coefficients. They are therefore *like quantities*, and may be added. Adding their coefficients (sect. 22) we get $a + b - 1$, which we inclose in a bracket, because they are now considered as only one quantity, a compound coefficient of x. Similarly, the coefficients of y in $-by$, cy, and $-ay$, are $-b$, c, and $-a$, which, when added and bracketed, become $-a - b + c = -(a + b - c)$.

In (3) the coefficients of x are a and $-b$, and subtracting $-b$ from a, we get $a+b$; those of y are $-b$ and c, and subtracting c from $-b$, we get $-b-c=-(b+c)$.

In (5) the coefficients of x^2 are $-a$ and c, which, when added, become $-a+c=-(a-c)$; those of x are $-ac$ and $-b$, which, when added, give $-ac-b=-(ac+b)$.

In (6) the first term of the quotient is $-a^2$, which, multiplied by the divisor, $a-b-c$ or $a-(b+c)$, gives $-a^3+a^2b+a^2c$ $=-a^3+a^2(b+c)$. In the second multiplication, $(b+c)(b+c)$ $=(b^2+2bc+c^2)$, by formula 1, sect. 33, or by actual multiplication; and as $(b+c)$ is negative in both divisor and quotient, the product is $+$: in the third, $(b^2-bc+c^2)(b+c)=(b^3+c^3)$, by formula 5. Also, in subtracting, $3abc-a(b^2+2bc+c^2)$ $=a(3bc-b^2-2bc-c^2)=a(-b^2+bc-c^2)=-a(b^2-bc+c^2)$.

The safest way of working examples like (2) and (4) is to clear away brackets, and bracket the answer when found.

Example (6) is the same as the second sum in Exercise XVIII., 35. The ordinary way of working it is perhaps as safe as the one given here, but it is twice as long and not so neat.

Examples:

Addition.

(1.)

$$\begin{array}{l} ax-by+3cz \\ bx+cy-az \\ -\ x-ay+bz \\ \hline (a+b-1)x-(a+b-c)y \\ \qquad -(a-b-3c)z \end{array}$$

(2.)

$$\begin{array}{r} -(a-b)x+5(a-c)y \\ (a-c)x-(a+c)y \\ -(a+b)x-(a-5c)y \\ \hline -(a+c)x+(3a-c)y \end{array}$$

Subtraction.

(3.)

$$\begin{array}{l} ax-by+cz \\ -bx+cy-az \\ \hline (a+b)x-(b+c)y+(a+c)z \end{array}$$

(4.)

$$\begin{array}{r} 3(a+b)x-2(b-c)y \\ (a-2b)x-(b+c)y \\ \hline (2a+5b)x-(b-3c)y \end{array}$$

(5.) *Multiplication.*

$$\begin{array}{l} x^2-ax-b \\ x+c \\ \hline x^3-ax^2-bx \\ \qquad cx^2-acx-bc \\ \hline x^3-(a-c)x^2-(ac+b)x-bc. \end{array}$$

(6.) *Division.*

$$
\begin{array}{l}
a-(b+c)\big)\,-a^3+3abc+(b^3+c^3)\big(-a^2-a(b+c)-(b^2-bc+c^2) \\
\quad\underline{-a^3+a^2(b+c)\qquad\qquad} \\
\qquad\quad -a^2(b+c)+3abc \\
\qquad\quad \underline{-a^2(b+c)+a(b^2+2bc+c^2)} \\
\qquad\qquad\qquad -a(b^2-bc+c^2)+(b^3+c^3) \\
\qquad\qquad\qquad -a(b^2-bc+c^2)+(b^3+c^3)
\end{array}
$$

Ans. $-a^2-a(b+c)-(b^2-bc+c^2)=-a^2-ab-ac-b^2+bc-c^2$.

Exercise XXII.

Addition.

1. $ax+by,\ bx+cy,\ -cx-dy$. 2. $ax-by,\ -bx+cy,\ cx-ay$.
3. $ax-by,\ y-bx,\ x-cy$. 4. $bx-3cz,\ bz-2ax,\ cx-az$.
5. $ax^2-bxy+y^2,\ -x^2+axy-by^2,\ cx^2-xy+cy^2$.
6. $a^2x-5xyz+4y,\ -2b^2x-axyz+by,\ 3c^2x-2bxyz-ay$.
7. $ax+by+cz,\ -bx-cy+az,\ cx-ay-bz$.
8. $(a+b)x-(a-c)y,\ (a-b)x+(a+c)y,\ (b-c)y-(a-c)x$.
9. $(a-b)x-2(b-c)y,\ 2(b+c)y-(a+b)x,\ 3(a+b)x-(b+c)y$.

Subtraction (the minuend coming first in each sum).

1. $ax+by,\ bx-ay$. 2. $ax-by,\ -bx-ay$.
3. $ax+by,\ -bx+ay$. 4. $ax+by,\ bx+ay$.
5. $ax-y,\ ay-x$. 6. $x-ay,\ y-ax$. 7. $by-ax,\ bx-ay$.
8. $ay-bx,\ ax-by$. 9. $-bx-ay,\ ax+by$.
10. $ax+by+cz,\ bx-cy-az$. 11. $ax+by+cz,\ bx+cy+az$.
12. $ax+by+az,\ bz-ay-bx$. 13. $ax+by-az,\ bx+ay-bz$.
14. $(a+b)x+(c+d)y,\ (a-b)x+(c-d)y$.
15. $(a-b)x-(c+d)y,\ -(a+b)x-(c-d)y$.
16. From $3(a+b)x+3(c+d)y$ take $2(a-b)x-2(c-d)y$ and $-2(a-b)x+3(3c-d)y$.

Multiplication.

1. $(x+m)(x+n),\ (y-m)(y-n),\ (xy+m)(xy-n)$.
2. $(x^2-a)(x^2+b),\ (px+q)(px-r),\ (ax+by)(cx+dy)$.
3. $(x^2+ax+c)(x+b),\ (x^2+ax+c)(x-b),\ (x^2-ax-c)(x+b)$.

4. $(x^2 - ax - 1)(x - 1)$, $(x^2 - ax + b)(x^2 + mx - n)$.
5. $(x^2 + x - b)(x^2 - mx + n)$, $(x^2 - mx + n)(x^2 + x - 1)$.
6. $(ax^2 - bx + c)(ax^2 + bx + c)$, $(y^2 - my + n)(y^2 + py - q)$.
7. $(y^2 - ayz + bz^2)(y^2 + cyz + dz^2)$, $(z^3 - az^2 - bz + c)$ $(z^2 - qz + r)$.
8. $\{x^2 + (m + n)^2x + (m + n)\}\{x + (m + n)\}$.
9. $\{y^3 - (a - b)^3y^2 + (a - b)^2y + a - b\}\{y - a + b\}$.
10. $\{x^2 - (a - b)x - ab\}\{x^2 + (a + b)x + ab\}$.
11. $\{y^2 - (a - c)y - ac\}\{y^2 + (b - d)y - bd\}$.
12. $(x^3 - mx^2 + nx - p)(x^2 - x + 1)$, $(x^3 - mx^2 - nx + 1)$ $(x^2 + x + 1)$.

Division.

1. $x^2 + (m + n)x + mn$	by $x + m$.
2. $x^2 + (a - bc)x - abc$	$a - bc$.
3. $y^2 - (3a - 2b)y - 6ab$	$y - 3a$.
4. $(a^2 - b^2)x^4 + (a^2 - 2ab + b^2)x^3 - (a^2 - ab)x^2$	$(a - b)x^2$.
5. $(m^3 - n^3)y^3 - m(m^2 - n^2)y^2 + n(m - n)^2y$	$(m - n)y$.
6. $(p + q)^2z^5 + 2(p^2 + pq)z^3 - 3(p^2 - q^2)z$	$(p + q)z$.
7. $z^2 + (a - 1)z - (2a^2 + 5a + 3)$	$z - (a + 2)$.
8. $x^3 + (a + c)x^2 + (ac + b)x + bc$	$x + c$.
9. $x^3 + (a - c)x^2 - (ac - b)x - bc$	$x - c$.
10. $x^3 - (a - c)x^2 - (ac + b)x - bc$	$x + c$.
11. $x^3 - (a - 1)x^2 - (a - 1)x + 1$	$x + 1$.
12. $x^3 - (a + p)x^2 + (q + ap)x - aq$	$x - a$.
13. $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$	$x - c$.
14. $x^3 + 3px^2 + (3p^2 - q^2)x + p(p^2 - q^2)$	$x + p - q$.

15. $a(a + 1)x^2 - (a^2 - 2a - 1)x - (a - 1)$ by $(a + 1)x - (a - 1)$.
16. $x^4 - a^2x^2 - b^2x^2 + a^2b^2$ by $x^2 - ax - bx + ab$.
17. $x^4 + (a - m)x^3 - (am + b - n)x^2 + (an + bm)x - bn$ by $x^2 - mx + n$.
18. $y^4 + (a - 2)y^3 - (2 + 2a - b)y^2 + (4 + ab)y - 2b$ by $y^2 + ay - 2$.
19. $amx^4 - (bm - an)x^3 + (cm - bn - ap)x^2 + (cn + bp)x - cp$ by $mx^2 + nx - p$.
20. $x + (m - 1)x^3 - (2m + 1)x^2 + (m^2 + 4m - 5)x + 3m + 6$ by $x^2 - 3x + m + 2$.
21. Work by this method Exercise XVIII. 36.

The following formulæ should be committed to memory:

I.—FORMULÆ FOR MULTIPLICATION.

1. $(x+a)(x+b)=x^2+(a+b)x+ab.$
2. $(x-a)(x-b)=x^2-(a+b)x+ab.$
3. $(x+a)(x-b)=x^2+(a-b)x-ab.$
4. $(x-a)(x+b)=x^2-(a-b)x-ab.$
5. $(x+a)^2=x^2+2ax+a^2,\quad \therefore (x+1)^2=x^2+2x+1.$
6. $(x-a)^2=x^2-2ax+a^2,\quad \therefore (x-1)^2=x^2-2x+1.$
7. $(x+a)(x-a)=x^2-a^2,\quad \therefore (x+1)(x-1)=x^2-1.$
8. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$
9. $(x+a)^3=x^3+3ax^2+3a^2x+a^3,\ \therefore (x+1)^3=x^3+3x^2+3x+1.$
10. $(x-a)^3=x^3-3ax^2+3a^2x-a^3,\ \therefore (x-1)^3=x^3-3x^2+3x-1.$

II.—FORMULÆ FOR RESOLUTION INTO FACTORS.

1. $x^2+(a+b)x+ab=(x+a)(x+b).$
2. $x^2+2ax+a^2=(x+a)(x+a),\ \therefore x^2+2x+1=(x+1)(x+1).$
3. $x^2-2ax+a^2=(x-a)(x-a),\ \therefore x^2-2x+1=(x-1)(x-1).$
4. $x^2-a^2=(x+a)(x-a),\quad \therefore x^2-1=(x+1)(x-1).$
5. $x^3+a^3=(x+a)(x^2-ax+a^2),\ \therefore x^3+1=(x+1)(x^2-x+1).$
6. $x^3-a^3=(x-a)(x^2+ax+a^2),\ \therefore x^3-1=(x-1)(x^2+x+1).$
7. $x^3+3ax^2+3a^2x+a^3=(x+a)^3,\ \therefore x^3+3x^2+3x+1=(x+1)^3.$
8. $x^3-3ax^2+3a^2x-a^3=(x-a)^3,\ \therefore x^3-3x^2+3x-1=(x-1)^3.$

TEST EXERCISES.*

I.

1. If $x = 6$, $y = 4$, $z = 2$, find the numerical values of
$(x + y)(x + z)$; $(x - y + z)^2$; xyz^2; $\sqrt{2x + y} - 2z$.
2. From the sum of $3x^2 - 7x + 5$, $x^2 + 6x - 4$, $4x^2 - 3x$, $7x - 3$, $-4x^2 - 3x + 1$ subtract $3x^2 - 5x + 7$.
3. Multiply $3a^4 + 7a^3 + a^2 - 7a - 4$ by $3a^2 - 7a + 4$.
4. Divide $x^5 - 2x^4 - 4x^3 + 19x^2 - 31x + 15$ by $x^2 - 7x + 5$.

II.

1. If $a = 3$, $b = 2$, $c = 1$, $d = 0$, find the values of
$a^2 + b^2 + c^2 - ab - ac - bc$; $a^2b^2c^2d^2 + abc - bcd$.
2. Simplify $a + b - [c + a - \{b + c - (a + b - \overline{c + a} - b)\}]$.
3. Multiply $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
4. Divide $1 - 9x^8 - 8x^9$ by $1 + 2x + x^2$.

III.

1. When $a = 6$, $b = 5$, $c = 4$, find the value of
$\dfrac{ab + ac + bc}{abc}$; $\dfrac{abc}{a^2 + b^2 + c^2}$; $\sqrt{2ab + c}$.
2. Add together $3a^3 - 5b^3 - 7c^3$, $4b^2 + c^3 - 2a^3$, $5c^3 + a^3 - 3b^3$, $9a^2 - 11b^3 - 13c^3$, $3b^2 - 4a^3 + 11c^2$.
3. Find the continued product of $(x^2 + x + 1)(x^2 - x + 1)(x^2 - 1)$.
4. Divide $a^6 - a^4b^2 + a^2b^4 - b^6$ by $a^2 - b^2$.

IV.

1. If $a = 5$, $b = 3$, show that $\dfrac{a^3 - b^3}{a - b} = a^2 + ab + b^2$.
2. Take $3x^4 - 2x^3 + 5x^2 - 2x + 1$ *twice* from $7x^4 - 4x^3 + 3x^2 + 5x - 1$.
3. Multiply $x^2 + y^2 - xy + x + y - 1$ by $x + y - 1$.
4. Divide $4y^4 + 81$ by $2y^2 + 6y + 9$.

* Exercises XIII.–XXIV. contain an additional question, illustrative of *Factorising* and *General Results in Division*, which may be omitted if necessary.

V.

1. If $a = 2$, $b = -3$, find the value of
$$\frac{a^2+ab+b^2}{a^2-ab+b^2}+\frac{4a+b}{5a-3b}+\frac{2a-b}{3a-b}.$$
2. Simplify $(a+b+c)(a+b-c)-(a-b+c)(a+b-c)$.
3. Multiply together $3-y$, $1-y$, $y+3$, $y+1$.
4. Divide $(x^2+y^2)^2+4(x^2+y^2)+4$ by x^2+y^2+2.

VI.

1. Evaluate $\dfrac{d-a}{d+a}+\dfrac{d+c}{d-c}-2\dfrac{d+b}{d-b}$,
when $a=1$, $b=2$, $c=3$, $d=4$.
2. Find the sum of $7a(x-y)+5b(y-z)+xy$, $a(x-y)+3xy$, $3a(x-y)-7xy$ and $4xy-11a(x-y)-6b(y-z)$.
3. Multiply $3m^2+2mn+n^2$ by $m^2-2mn+3n^2$, and prove the result.
4. Divide $ab+2a^2-3b^2+4bc+ac-c^2$ by $a-b+c$.

VII.

1. What is the value of $\dfrac{x+2\sqrt{xy}+y}{\sqrt{x}+\sqrt{y}}+\dfrac{x-2\sqrt{xy}+y}{\sqrt{x}-\sqrt{y}}$,
if $x=16$ and $y=9$.
2. Add together $\frac{1}{2}(2a-3b+5c)$, $\frac{1}{3}(3c-2a+5b)$,
$\frac{1}{4}(5b-4c+3a)$, $\frac{1}{5}(5c-3a-4b)$.
3. Multiply $a^2+4b^2+9c^2+2ab+3ac-6bc$ by $a-2b-3c$.
4. Divide $3x^4-8x^3y-4x^2+4x^2y^2-4xy-15$ by $3x^2-2xy+5$.

VIII.

1. Find the value of $\dfrac{x^4-81}{x^3+3x^2+9x+27}$, when $x=4$.
2. Take the sum of the *first* three of the following expressions from the sum of the *last* three:
$a^3+3a^2b+2ab^2+b^3$, $2a^3+a^2b-3ab^2-2b^3$, $3a^2b-4ab^2$,
$4a^3-2a^2b+3ab^2-b^3$.
3. Expand $(x+y)^2(x-y)^2(x^2+y^2)^2$.
4. Divide $a^8+a^4b^4+b^8$ by $a^4+a^2b^2+b^4$.

IX.

1. If $a = 5$, $b = 3$, find the value of
$$3a - [a - b - \{a + b - c + d - (a + b - c + d)\}].$$
2. Simplify $(x + 3)(x + 4) + (x - 2)(x - 3) - (x + 1)(x - 2)$.
3. Find the continued product of $a - bc$, $a + bc$, $a^2 + b^2c^2$.
4. Divide $a^5 + ab^4 - b^5$ by $a^2 - ab + b^2$.

X.

1. Find the value of $(n^2 + n + 1)^2 + (n^2 - n + 1)^2$, when $n = -5$.
2. Add together $3a - 4b + 5c - 2d$, $3a + 5c$, $2b - c + 3d$, $a + 3b - 4d$, and subtract the sum from $8a - 7b - 6c + 5d$.
3. Find the square of $x^2 - 6xy + 9y^2$.
4. Divide $a^5b^5 - 3a^2b^2 + 2$ by $ab - 1$.

XI.

1. Find the value of $\sqrt{m^2 + 2np} + \sqrt{n^2 + mp} + \sqrt{p^2 + mn}$ when $m = 4$, $n = 3$, and $p = -2$.
2. Simplify $x - [2y + \{3z - 2x - (5y - 3z)\} - \{2x - (5y - 3z)\}]$.
3. Multiply $x^3 - 4x^2 + 2x - 3$ by $2x^3 - 3$.
4. Divide $z^4 + 64a^4$ by $z^2 + 4a + 8a^2$.

XII.

1. If $x = 2$, $y = 3$, $z = -5$, find the numerical value of (1) $x^2 + y^2 + z^2$; (2) $x^3 + y^3 + z^3 - 3xyz$.
2. Simplify $\{a + (b - c)\} - \{(a - b) + c\} + \{a - (b + c)\}$.
3. Find the continued product of
$$(x + 2y)(x - 2y)(x^4 + 4x^2y^2 + 16y^4).$$
4. Divide $a^2 - 2ab + b^2 - c^2 + 2a - 2b - 2c$ by $a - b - c$.

XIII.

1. Find the value of $(a - b)^2 + (b - c)^2 + (c - a)^2$, and of $(a + b)(a - b) + (b + c)(b - c) + (c + a)(c - a)$, when (1) $a = 4$, $b = 2$, $c = 1$; (2) $a = b$, and $c = 0$.
2. Simplify $3[8x - \{6y + (4x - 3y) - (2x + 3y)\}]$.
3. Multiply the sum of $a^2 + 6ab + 9b^2$ and $a^2 - 6ab + 9b^2$ by the difference between $a^2 - 2ab + b^2$ and $a^2 + 2ab - 17b^2$.
4. Divide $a^6 + 2a^3b^3 + b^6$ by $a^2 + 2ab + b^2$.
5. Write down the following products:
$(a - 3)(a - 2)(a + 5)$; $(x + m)(x + n)(x - m)(x - n)$;
$(y^2 + y + 1)(y^2 + y - 1)$; $(a + 2b)^2(a - 2b)^2$.

XIV.

1. Find the sum of $(a+2d)^2+(a+d)^2+a^2+(a-d)^2+(a-2d)^2$.
2. From $3a(b-c)x-2l(m+n)y$ take $a(b-c)x+2l(m+n)y$.
3. Multiply together $3ab(9a^2+b^2)(3a+b)(3a-b)$.
4. Divide $x^4-x^2y^2-x^2z^2+y^2z^2$ by $x^2+xy-xz-yz$.
5. Find the sum of the following quotients:

$$\frac{a^3+b^3}{a+b}, \quad \frac{a^3-b^3}{a-b}, \quad \frac{a^3+8b^3}{a+2b}, \quad \frac{a^3-8b^3}{a-2b}.$$

XV.

1. Find the value of $\dfrac{(a+b)^2-(c+d)^2}{(a+b)-(c+d)}$, when $a=4$, $b=3$, $c=2$, $d=1$.
2. From $a+b+c$ take the sum of $\frac{1}{2}a+\frac{1}{3}b+\frac{1}{4}c$, $\frac{3}{4}a+\frac{1}{2}c$, $-\frac{5}{8}a+\frac{1}{6}b-\frac{3}{4}c$, and $\frac{7}{8}a+\frac{1}{2}b+\frac{3}{4}c$.
3. Add together $(m+n+1)^2+(m+n-1)^2+(m-n+1)^2+(-m+n+1)^2$.
4. Divide $a^4x^4+a^2b^2x^2y^2+b^4y^4$ by $a^2x^2+abxy+b^2y^2$.
5. Resolve x^4-7x^2+12; $1-16x^4$; $a^2-6ab+9b^2$.

XVI.

1. Add together $x(x+y+z)$, $y(x+y-z)$, $z(x-y+z)$, and find the value of the sum when $x=y=z=2$.
2. From $(a-b)xy+2(b-c)yz+3(c-a)zx$ take $axy+2byz+3cxz$.
3. Write down the squares of

   $1-3x$, $2m+5m$; $pq-3$, r^2-4s^2, $x-3yz$.
4. Divide x^8y^8-1 by $x^6y^6+x^4y^4+x^2y^2+1$.
5. Simplify $\dfrac{x^2-5x+6}{x-2}-\dfrac{x^2-x-42}{x-7}$.

XVII.

1. Find the difference between a^4-b^4 and $(a-b)^4$, when $a=5$, $b=3$.
2. Collect the coefficients of like powers of x in

   $ax^2-2abx+a-2bx^2+3acx+2b+3cx^2-6bcx+3c$.
3. Simplify $(a^2-1)(a^4+a^2+1)-(a^2+1)(a^4-a^2+1)$.
4. Divide a^8-b^8 by $a^3+a^2b+ab^2+b^3$.
5. Resolve $2mx-my-4nx+2ny$, $27x^3-1$, $25a^2b^2-10ab+1$.

XVIII.

1. Find the value of $(ax + by)^2 + (by + cz)^2 + (cz + ax)^2$, when $a = 4$, $b = 2$, $c = 1$, $x = \frac{1}{4}$, $y = \frac{1}{2}$, $z = 1$.
2. Simplify $x^4 - [4x^3 - \{6x^2 - (4x - 1)\}] - (x^4 - 4x^3 + 6x^2 - 4x + 1)$.
3. Show that
$$(x-y)(x+y-z)+(y-z)(y+z-x)+(z-x)(z+x-y)=0.$$
4. Divide $19x^2 - 2x^4 + 15 - 31x - 4x^3 + x^5$ by $5 + x^3 - 7x$.
5. Add together the following quotients:
$$\frac{x^2+2x+1}{x+1},\ \frac{x^4-1}{x-1},\ \frac{x^3+3x^2+3x+1}{x+1},\ \frac{x^2-1}{x+1}.$$

XIX.

1. If $a = 4$, $b = 5$, $c = 1$, $m = 2$, $n = 3$, find the value of $(a^m + b^n) + (b^m + c^n) + (c^m + a^n)$.
2. Add together $a^3 - \{3a^2 - (3a - 1)\}$ and $a^3 + \{3a^2 - (3a + 1)\}$.
3. Multiply $x^3 - x^2 + x - 1$ by $x^3 + x^2 + x + 1$, and divide the product by $x^4 + 2x^2 + 1$.
4. Divide $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$ by $a + b - c - d$.
5. Resolve into factors:
$$x^3 - 2x^2 - 24x,\ x^7 - xy^6,\ x^2 + 2xy + y^2 - z^2.$$

XX.

1. Add together:
$$(2b+2c - a)x^2 + (2c + 2a - b)x + (2a + 2b - c).$$
$$(3b + 3c - 2a)x^2 + (3c + 3a - 2b)x + (3a + 3b - 2c).$$
$$(4b + 4c - 3a)x^2 + (4c + 4a - 3b)x + (4a + 4b - 3c).$$
2. Simplify $(2a - b)(2a + b) - [ab - b\{a - (2a - \overline{2a - b})\}]$.
3. Multiply $y^3 - 2y^2 + 3y - 4$ by $y^3 + 2y^2 + 3y + 4$.
4. Divide $27(a - b)^4 - 18(a - b)^3 + 9(a - b)^2$ by $9(a - b)$.
5. Resolve $x^2 - 6x + 9$, $a^3 - 4a^2b + 4ab^2$, $m^3n^3 - 125$.

XXI.

1. Find the value of $7 - \sqrt{9 + x^2} + \frac{1}{3}(x^3 + 5x^2 + 2x - 8)$, when $x = 4$, -4 and 0.
2. Express with brackets taking the terms (1) *two* (2) *three* together: $ax - by - cz - dx + ey - fz$.
3. Multiply $x + \frac{1}{2}y + \frac{1}{3}z$ by $x - \frac{1}{2}y - \frac{1}{3}z$.

4. Simplify $\dfrac{(x^3 - 2x^2 - 4x + 8)(x^2 - 2x + 4)}{x^3 + 8}$.

5. Prove $(a + 4b)^2 - (a - 4b)^2 - (4ab)^2 = 16ab(1 - ab)$.

XXII.

1. Evaluate the following expressions when $a = 9$, $x = 7$.

 (1) $14a - [a - 5\{a + 3(a - x)\}]$; (2) $\sqrt{\dfrac{a^3 - x^3 - 3ax(a - x)}{a - x}}$.

2. From $5\{(a - b) - (c + d)\}$ take $2\{2(a + b) - 3(c - d)\}$.

3. Write down the square and the cube of

 (1) $x - 2a$, (2) $3mn + 1$.

4. Divide $24b^4 + 43a^2b^2 - 38ab^3 - 22a^3b + 8a^4$ by $4a^2 - 5ab + 6b^2$.

5. Add together the following quotients:

 $\dfrac{9m^2 - n^2}{3m - n}$, $\dfrac{m^2 + 5mn + 6n^2}{m + 2n}$, $\dfrac{m^2 - 16n^2}{m + 4n}$, $\dfrac{m^2 - mn - 12n^2}{m - 4n}$.

XXIII.

1. If $a = 3$, $b = 1$, find the numerical value of

 $(\sqrt{5a + b})(\sqrt{a - 2b}) + (\sqrt{9a - 2b})(\sqrt{a - 3b})$.

2. Add together $a - \frac{3}{4}b + \frac{1}{5}c$, $\frac{1}{3}a + \frac{1}{6}b + \frac{2}{3}c$, $\frac{1}{3}b - \frac{8}{15}c$.

3. Simplify $(x^2 + xy + y^2)^2 + (x^2 - xy + y^2)^2 - 2(x^4 + x^2y^2 + y^4)$.

4. Divide $(x + 1)^6 + 3(x + 1)^4 + 3(x + 1)^2 + 1$ by $(x + 1)^2 + 1$.

5. Resolve $x^2 - 5xy - 84y^2$, $a^4 - ax^3$, $y^2 + 2yz + z^2 - 1$.

XXIV.

1. Remove brackets from

 $ax^3 - [bx^2 + cx - \{bx^3 + cx^2 - (dx + e)\}]$,

 and regroup the terms according to powers of x.

2. From $3\{a - 2b - (3c + 4d)\}$ take $2\{2a + b - (c - 3d)\}$.

3. Given quotient $= x^2 - 5x + 8$, and divisor $= 3 - x$, to find the dividend. Verify the result when $x = 1$.

4. Divide $(ab + 1)^2(a^2b^2 - ab + 1) - (a^2b^2 - 1)(a^4b^4 + a^2b^2 + 1)$ by $a^3b^3 + 1$.

5 Show that $x(x + 1)(x + 2)(x + 3) + 1 = (x^2 + 3x + 1)^2$.

STANDARD ALGEBRA.—PART II.

SIMPLE EQUATIONS.

44. AN EQUATION is a statement that two quantities are equal (L. *æquus*, equal).

The term *Equation* is, however, usually restricted to cases in which the quantities are equal only when the letters they contain have each a particular value. Thus, $x + 5 = 9$ is an *equation*, the statement being true only when $x = 4$.

If the quantities are always equal, no matter what values their letters may have, the equation is called an *Identity;* as in

$$(a + b)^2 = a^2 + 2ab + b^2, \quad x^2 - y^2 = (x + y)(x - y).$$

45. *A Simple Equation*, or *Equation of the First Degree*, is one which, when reduced to a simple form, contains only the *first power* of the unknown quantity.

46. *To solve an Equation* is to find the value of the unknown quantity, or quantities, it contains.

The value of the unknown quantity is called the *Root* of the equation.

47. The solution of equations is founded on the following *Axioms:*

(1.) If equals be added to equals, the sums are equal.

If $x - a = b$; then $x - a + a = b + a$; that is, $x = b + a$.

(2.) If equals be taken from equals, the remainders are equal. If $x + a = c$; then $x + a - a = c - a$; that is, $x = c - a$.

Similarly, if $2x - 4 = x + 2$, then $2x - x = 2 + 4$.

(3.) If equals be multiplied by equals, the products are equal.

(4.) If equals be divided by equals, the quotients are equal.

If $\frac{x}{3} + \frac{x}{4} = 7$; then, by Ax. 3, $\frac{12x}{3} + \frac{12x}{4} = 7 \times 12$; that is, $4x + 3x = 84$; that is, $7x = 84$; and therefore, by Ax. 4, $x = 12$.

48. 1. Any term of an equation may be transposed from one side of the equation to the other if its sign be changed (Axioms 1, 2).

2. If every term of an equation be multiplied, or divided, by the same quantity, the equality is not altered (Axioms 3, 4).

This enables us to *clear an equation of fractions.*

3. If the sign of every term of an equation be changed, the equality is not altered; for every term may be multiplied by -1. Thus, if $-2x + 6 = -x + 1$, then $2x - 6 = x - 1$.

4. The position of the two sides of an equation may be changed. Thus, if $7 = x - 5$, then $x - 5 = 7$.

Solution of Simple Equations containing One Unknown Quantity.

49. (a) Rule.—(1) Transpose the terms containing the unknown quantity to one side, and the known quantities to the other; (2) collect the terms on each side into one sum; (3) divide both sides by the coefficient of the unknown quantity.

Note.—The working of every example should be tested by putting the value found for x in the place of x.

Examples.

1. Solve $5x - 3 = 4x + 6$.

Transposing, $5x - 4x = 6 + 3$

Collecting, $x = 9$

Proof, $(5 \times 9) - 3 = (4 \times 9) + 6$

that is, $42 = 42$

2. Solve $2x + 5 = 25 - 3x$.

Transposing, $2x + 3x = 25 - 5$

Collecting, $5x = 20$

Dividing by 5, $x = 4$

3. Solve $4x + 6 = 9x - 4$.

Transposing, $4x - 9x = -4 - 6$
Collecting, $-5x = -10$
Changing signs, $5x = 10$
Dividing by 5, $x = 2$

4. Solve $9x - 2 = 3x + 2$

Transposing, $9x - 3x = 2 + 2$
Collecting, $6x = 4$
Dividing by 6, $x = \frac{4}{6}$ or $\frac{2}{3}$

5. Solve $5(x+4) - 2(x-5) = 0$
that is, $5x + 20 - 2x + 10 = 0$
Transposing, $3x = -20 - 10$
$3x = -30$
Dividing by 3, $x = -10$

6. Solve $a(x - b) = c(x - d)$
that is, $ax - ab = cx - cd$
Transposing, $ax - cx = ab - cd$
$(a - c)x = ab - cd$
Dividing by $a - c$, $x = \frac{ab - cd}{a - c}$

Exercise XXIII. (A).

1. $x + 5 = 8$; $x - 7 = 3$; $x - 9 = 0$; $x + 8 = 0$; $4x + 4 = 3x + 7$.
2. $5x - 8 = 4x - 3$; $8x + 5 = 5x + 26$; $9x - 7 = 4x + 23$.
3. $5x - 3 = 6x - 7$; $8x - 11 = 9x - 17$; $5x + 11 = 7x - 9$.
4. $11x - 6 = 15x - 18$; $38 - 6x = x + 3$; $6x - 7 = 59 - 5x$.
5. $85 - 9x = 45 - 4x$; $12x - 7 = 9x - 5$; $8x + 5 = 24x - 5$.
6. $13x + 6 = 8x - 9$; $48 + 5x = 7x + 62$; $9 - 8x = 61 + 8x$.
7. $4(x+3) = 3(x+6)$; $4(x+5) = 5(8-x)$; $5(x-8) = 7(12+x)$.
8. $5x - 9x + 23 = 8x - 2x - 47$; $13x - 8x + 12 = 7x + 4x - 12$.
9. $3x - 13x + 33 = 3x - 8x - 27$; $5x - 6x + 28 = x - 4x - 10$.
10. $2x + 5(x+4) = 7x + 5(x-4)$; $10x + 7(x-3) = 3x + 7(x+7)$.
11. $5x - (2 + 3x) = 7x - (3x - 8)$; $2x - 3(4 - x) = 5x - 2(x - 3)$.
12. $(x+3)(x+6) = (x+1)(x+9)$; $(x-5)(x+7) = (x-2)(x+1)$.
13. $(x+9)(x-9) = (x-1)(x-7)$; $(x-5)^2 = (x-9)(x+3)$.
14. $(x+9)^2 = (x+8)^2 + 4x + 7$; $(x+5)^2 - (x-1)^2 = 15x$.
15. $x - 3(4 - x) = 3x - 4(x+2)$; $9x - 5(x-4) = 4x - 5(7 - x)$.
16. $3x + 5(9 - x) - (x + 3) = 0$; $15x - 7(3x - 4) + 4(2x - 3) = 0$.
17. $16 - \{4x - 2(8 - 2x)\} = 0$; $x - [2x - \{5x - 4(8 - 3x)\}] = 0$.
18. $(x - 8)(x + 5) + (x + 9)(x - 4) - 2(x - 3)(x + 4) = 2 - 3x$.
19. $(3x - 4)(x + 5) - (x - 7)(8 + 3x) = 18(x + 8) + x + 2$.
20. $(2x - 3)(3x - 2) - 4x(x - 1) + 4 = (2x + 1)(x - 4) - 6$.
21. $(x - 5)^2 - (5 - x)^2 + 10x(x - 2) = (5x - 8)(2x - 1)$.
22. $53 + 5(x + 5)(x - 6) - (3x + 4)(3x - 4) = 12 - (2x + 3)^2$.

23. $6x + 4 : 11x - 4 :: 2 : 3$; $2(x + 3) : 6 :: 5(x - 1) : 7$.
24. $3 : 10 :: 15x - (1 + 5x) : 10x + 3(4 - 5x)$.
25. $cx = a + b$; $rx = 2p - qx$; $mx - 1 = x + m - 2$.
26. $ax - b = ab - x$; $ax + c = bx + d$; $mx - p = nx + q$.
27. $2ax + bx = a(a + x) - b^2$; $a(x - bc) + 2bx = b(x + bc)$.
28. $m(x - m + 2p) + n^2 = nx + 2mp$; $p(qx + c) - ab = c(p + 1) - rx$.
29. $a(x + 2) + b(x + 3) = 2c(2 + x) + b$; $(a + b)^2 x = c^2 + 4abx$.
30. $p(x - p) - q(p + q) = 3q(p - x) + 2q(q + x)$.

Problems Resulting in Simple Equations.

Examples, with Notes.

49. (b) *Note* 1.—The whole *minus* one part = the other part.

1. Divide 24 into two parts, one of which shall exceed the other by 6.

(1.) Let x = the greater part,	Or, (2.) let x = the less,
then $24 - x$ = the less.	then $24 - x$ = the greater.
And $x - 6 = 24 - x$,	And $x + 6 = 24 - x$,
or $x = (24 - x) + 6$.	or $x = (24 - x) - 6$.

A problem may often be set down algebraically in more ways than one. By any of the four given above, we find the greater part to be 15, the less 9.

Note 2.—If four quantities be proportional, the product of the extremes is equal to the product of the means.

Example.—If $x : y :: 2 : 3$, or $\frac{x}{y} = \frac{2}{3}$; then $3x = 2y$.

2. Divide 33 into two parts, so that the one shall bear the same proportion to the other as 5 bears to 6.

Let x = the one part, then $33 - x$ = the other.

Then $x : 33 - x = 5 : 6$; that is, $6x = 5(33 - x)$.

Or $33 - x : x = 5 : 6$; that is, $6(33 - x) = 5x$.

Or $\frac{x}{33 - x} = \frac{5}{6}$; Or $\frac{33 - x}{x} = \frac{5}{6}$.

From any one of these equations we find the parts to be 15 and 18. Had we said, let x = the greater, then we must have written $x : 33 - x = 6 : 5$, $33 - x : x = 5 : 6$, $\frac{x}{33 - x} = \frac{6}{5}$, or $\frac{33 - x}{x} = \frac{5}{6}$.

Note 3.—All the terms of an equation must be of the *same sort* (sect. 21). Thus, if one be expressed in pence, all must be in pence; if one be expressed in yards, all must be in yards, &c.

3. A person has £3, 7s. 6d. made up of half-crowns and half-sovereigns. He has 12 coins in all. How many has he of each?

Let $x =$ the No. of half-cr., then $5x =$ their value in sixpences;
$12-x=$ 〃 half-sov., and $20(12-x)=$ 〃 〃
£3, 7s. 6d. = 135 sixpences;
$\therefore 5x + 20(12-x) = 135$; whence $x = 7$, $12 - x = 5$.

Note 4.—It is often better to make a *multiple* of x stand for what is wanted.

4. A person spends £10 more than half his income on food and lodging, £2 less than half the remainder on clothes and books, and then has £12 left. Find his income.

Let $4x =$ income; $\because$ in taking $\frac{1}{2}$ of $\frac{1}{2}$ of $4x$ no fractions arise.

Then $2x + 10$ = expenditure on food, &c.

$\overline{2x - 10}$ = remainder.

$x - 5 - 2$ = expenditure on clothes, &c.

$\overline{x - 5} + 2 = £12$, whence $x = £15$, and $4x = £60$.

Exercise XXIII. (B).

1. (1) To twice a certain number 15 is added, and the result is 39; (2) to three times a number 8 is added, and the result is 50; (3) from five times a number 9 is taken, and the result is 41. Find the numbers.

2. A boy at play wins 15 marbles, and then has 4 times as many as he began with. How many had he at first?

3. A has £60, and B £20; A spends 5 times as much as B, and then they have equal sums. How much has each spent?

4. Divide (1) 37 into two parts, so that the first is 5 more than the second; (2) 42 into two parts, so that the first is 5 times the second; (3) 75 into three parts, so that the first shall be 6 more, and the third 9 less, than the second.

5. Find four consecutive numbers whose sum is 222.

6. Three horses cost £115. The first cost £15 more, and the third £5 less, than the second. Find the price of each.

7. Find (1) two numbers whose difference is 6, and of which 4

times the one equals 7 times the other; (2) two numbers whose sum is 28, and of which 5 times the one equals 9 times the other.

8. A and B have together £25, and three times A's money and four times B's would together make £85. How much has each?

9. Find (1) two numbers whose difference is 6, and their sum 54; (2) two numbers whose difference is 5, and of which 4 times the less exceeds 3 times the greater by 7.

10. A and B have together £60, and three times what A has would be £4 more than what B has. How much has each?

11. Find the area of an oblong whose sides are respectively 9 ft. greater and 6 ft. less than those of a square equal to it.

12. Two vessels sail from London; the one going 250 miles a day, sails 2 days before the other, which goes 275 miles a day. In how many days will the second overtake the first?

13. A starts from Edinburgh and B from Glasgow at the same time. A walks 4 miles an hour; B, 3. The distance between the two places is 42 miles. How far from Glasgow do they meet?

14. A farmer bought sheep at 17s. each, and had £2 over. Had they cost 22s. each, he would have had £2 too little. How many sheep were there, and how much money had he?

15. A farmer has 7 times as many sheep as cows, and 3 times as many cows as horses. They number 200 altogether. How many has he of each?

16. Find the ages of A and B (1) when A is three times as old as B, and is as much under 50 as B is under 20; (2) when A is 22 years older than B, and is as much over 40 as B is under 30.

17. Five brothers are each 3 years older than the one next him. The eldest is thrice as old as the youngest. Find their ages.

18. How much tea at 2s. and 3s. per lb. respectively must a grocer mix together so as to have 48 lbs. worth 2s. 5d. each?

19. A and B play at marbles. A begins with 16, B with 12. After the game A has thrice as many as B. How many has he won?

20. Divide (1) 30 into two parts, so that twice the one shall equal thrice the other; (2) 15 into two parts, so that the sum of 3 times the less and 4 times the greater may be 54.

21. How old are A and B (1) when A is twice as old as B, but 10 years since was 4 times as old as B; (2) when A is five times as old as B, but 6 years hence will be only 3 times as old as B?

22. Six years hence a boy will be 4 times as old as he was 6 years ago. How old is he?

23. A man aged 48 has 4 sons whose joint ages are 30. When will the joint ages of the sons equal their father's age?

24. Divide (1) 45 into two parts, so that the one may be to the other in the proportion of 2 to 3; (2) £64 between two persons, so that the one shall have £5 for every £3 the other has.

25. A person leaves £9000 to be divided between his wife, his 3 sons, and his 4 daughters. Each son is to get twice as much as each daughter, and the widow £1000 less than all the children together. How much does each get?

26. A sum of £6 is made up of crowns and florins, and there are 33 coins in all. How many are there of each?

27. A pays £7, 4s. in guineas and half-crowns, using 6 times as many half-crowns as guineas. How many of each are paid?

28. In a purse containing £9, 8s. there are three crowns for every sovereign, and 4 shillings for every crown, and there are no other coins. How many are there of each?

GREATEST COMMON MEASURE.

50. The *Greatest Common Measure* of two or more quantities is the highest factor contained in each of them.

Thus, the greatest common measure of $12x^2y$ and $16xy^2$ is $4xy$.

'Greatest Common Measure' (abbreviated G.C.M.) is the name generally used. 'Highest Common Factor' would be a better one (52, Note 5).

51. To find the G.C.M. of Simple and easily resolvable Compound Quantities. Resolve them into their elementary factors (sect. 7, 42); the product of the factors common to all of them is their G.C.M.

Examples.

(1.)

$$6a^3b^2 = 3 \,.\, 2 \,.\, a \,.\, a \,.\, a \,.\, b \,.\, b$$
$$9a^2b^3 = 3 \,.\, 3 \,.\, a \,.\, a \,.\, b \,.\, b \,.\, b$$
$$12a^2b^2 = 3 \,.\, 2 \,.\, 2 \,.\, a \,.\, a \,.\, b \,.\, b$$

(2.)

$$a^2 - ab = a(a-b)$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^2 - 2ab + b^2 = (a-b)(a-b)$$

(3.)

$$x^2 + 3x + 2 = (x+1)(x+2)$$
$$x^2 - x - 2 = (x+1)(x-2)$$
$$x^2 + 4x + 3 = (x+1)(x+3)$$

(4.)

$$x^2 + 4xy + 3y^2 = (x+y)(x+3y)$$
$$x^2 - 3xy - 4y^2 = (x+y)(x-4y)$$
$$x^2 - y^2 = (x+y)(x-y)$$

In (1), all the quantities contain the factors 3, aa, and bb; in (2), $a-b$; in (3), $x+1$; in (4), $x+y$. The G.C.M. of those in (1), is therefore $3a^2b^2$; in (2), $a-b$; in (3), $x+1$; in (4), $x+y$.

52. To find the G.C.M. of two Compound Quantities not easily resolvable. (1.) Arrange them according to powers of some letter common to both; (2.) Divide the one having the higher power of the leading letter by the other, or, if its highest power be the same in both, divide either by the other; (3.) Divide the divisor by the remainder; and so on, dividing, by each new remainder, the divisor immediately preceding it, till there is no remainder. The last divisor is the G.C.M.

Note 1. If the quantities have a *common* factor which can be seen by inspection, remove it from both before applying the rule, and keep it to be a factor of the G.C.M.

Note 2. Any divisor or dividend may be multiplied or divided by any number which does not introduce or remove a *common* factor. Therefore,

Note 3. If the first term of any divisor be negative, change the signs of all its terms; that is, multiply or divide it by -1. Doing so may change the signs of the G.C.M., but that (Note 5) does not matter.

Note 4. Where a large multiplier is required to make the one quantity divisible by the other, it sometimes happens that, if the order of the terms of each be reversed, a smaller multiplier or no multiplier at all is required; as in Exer. XXIV. 15, 17, 19, &c.

Note 5. The G.C.M. may have its signs changed, and still be the G.C.M.

Since $12x^2 = 4x \cdot 3x$ or $-4x \cdot -3x$, we see that factors may have their signs changed, and still be factors. Similarly, if $a - b$ is a factor of one or more quantities, $-(a-b)$, that is, $b - a$, will also be a factor of them; and either may be their G.C.M., for they contain the same powers of a and b. $a - b$ will be *greater* than $b - a$, if a is greater than b; but *less*, if b is greater than a.

In arithmetic, it is always the *greatest* common factor we seek; in algebra, the *highest*; that is, the one containing highest powers.

Examples.

1. Find the G.C.M. of $3x^4 - 9x^3 - 33x^2 + 15x$, and $4x^4 - 16x^3 - 26x^2 + 30x$.

We see that x is a factor of both; we therefore take it out, and keep it to be a factor of the G.C.M. (Note 1). We see, too, that 3 is a factor of the first quantity, and 2 of the second; we take them out also (Note 2), but as neither of them is a factor of both,

and therefore not a common factor, we do not keep them. The quantities now are $x^3 - 3x^2 - 11x + 5$, $2x^3 - 8x^2 - 13x + 15$.

$$x^3 - 3x^2 - 11x + 5)2x^3 - 8x^2 - 13x + 15(2$$
$$2x^3 - 6x^2 - 22x + 10$$
$$-2x^2 + 9x + 5 = -1(2x^2 - 9x - 5)$$

$$2x^2-9x-5)\ x^3-3x^2-11x+5(x+3$$
$$2$$
$$2x^3-6x^2-22x+10$$
$$2x^3-9x^2-5x$$
$$3x^2-17x+10$$
$$2$$
$$6x^2-34x+20$$
$$6x^2-27x-15$$
$$-7(x-5) = -7x+35$$

$$x-5)2x^2-9x-5(2x+1$$
$$2x^2-10x$$
$$x-5$$
$$x-5$$

G.C.M., $x(x-5)$, or $-x(x-5)$

Here we change the signs of $-2x^2 + 9x + 5$; that is, multiply or divide it by -1, before taking it for a divisor (Note 3). To make $x^3 - 3x^2 - 11x + 5$ divisible by $2x^2 - 9x - 5$, we multiply it by 2, which is not a factor of the latter, and cannot therefore be a factor of both (Note 2). For the same reasons we multiply $3x^2 - 17x + 10$ by 2, and divide $-7x + 35$ by -7 before taking it for a divisor.

The given quantities have thus two common factors, x and $x - 5$; their G.C.M. therefore is $x(x-5)$, or $-x(x-5)$; Note 5.

2. Find the G.C.M. of $x^3 - x^2y - xy^2 + y^3$, and $x^4 - x^3y - 2x^2y^2 + 2y^4$.

$$x^3 - x^2y - xy^2 + y^3)x^4 - x^3y - 2x^2y^2 + 2y^4(x$$
$$x^4 - x^3y - x^2y^2 + xy^3$$
$$-y^2(x^2 + xy - 2y^2) = -x^2y^2 - xy^3 + 2y^4$$

$$x^2+xy-2y^2)x^3-x^2y-xy^2+y^3(x-2y$$
$$x^3+x^2y-2xy^2$$
$$-2x^2y+xy^2+y^3$$
$$-2x^2y-2xy^2+4y^3$$
$$3y^2(x-y) = 3xy^2-3y^3$$

$$x-y)x^2+xy-2y^2(x+2y$$
$$x^2-xy$$
$$2xy-2y^2$$
$$2xy-2y^2$$

G.C.M., $x - y$ or $y - x$

Exercise XXIV.

Find, by inspection, or by sect. 59, the G.C.M. of

1. ab, bc; abc, bcd; x^3, x^2, x; x^2y, x^2z; $4x^2, 6x^3$; a^2b, ab^2.

2. $6xy^2, 9x^2y$; $4a^2x^3, 10a^3x^2$; $4a^3b^2, 6a^2b^3$; $18a^4x^5, 45a^6x^4$.

3. $16a^3x, 40a^2x^2, 56ax^3$; $a^2-x^2, a-x$; x^4-1, x^2+1.

4. $x^2-y^2, (x-y)^2$; $x^4-1, (x^2+1)^2$; $1-9x^2, (1-3x)^2$.

5. $x^3+y^3, (x+y)^2$; $x^3-1, (x-1)^3$; $x^6-y^6, (x^3+y^3)^2$.

6. $a^3-x^3, a^2-2ax+x^2$; x^3+1, x^2+2x+1; x^3-1, x^2+x+1; $1-x^3, 1-2x+x^2$; $1+x^5, (1+x)^5$.

7. $x^3+y^3, x^2-y^2, x^2+2xy+y^2$; x^3-1, x^2-1, x^2-2x+1.

8. $ab+b^2, a^2+2ab+b^2$; $a^2-2ax+x^2, ax-x^2, a^2-ax$.

9. $x^2+7x+12, x^2+9x+20$; $x^2-9x+20, x^2-8x+15$.

10. x^2+x-20, x^2-x-12; $x^2+8x+15, x^2-9$.

11. $x^2-3x-10, x^2-4$; $x^2-x-30, x^2+10x+25$.

12. $x^2-2x-15, 2x^2-3x-27$; $x^2+7x+12, 3x^2+22x+40$.

13. $4x^2+10x-24, 4x^2+14x-30$; $x^2-x-6, x^3-3x^2-2x+16$.

14. $x^2-5x-14, 3x^3-19x^2-10x-28$; $x^3+x^2-12x, x^4+3x^3-3x^2-45x$; $3x^2+11x+6, 2x^2+9x+9$.

15. $3x^2-11x-20, 4x^2-14x-30$; $5x^2-17x-12, 6x^2-21x-12$; $3x^2-25x+28, 4x^2-23x-35$.

16. $8x^3+2x^2-3x, 10x^3+9x^2-7x$; $4x^2+4x-15, 6x^2+3x-18$.

17. $6x^2-13x+5, 9x^2-3x-20$; $x^2y-xy^2-y^3, x^3y-2x^2y^2+y^4$.

18. $x^2-6xy+5y^2, x^3-6x^2y+4xy^2+y^3$; $4x^3-16x^2+7x+20, 4x^3+4x^2-33x-5$; $x^3-6x^2-2x+7, x^4-9x^3+14x^2+19x-25$.

19. $63x^3-25x^2-40x+12, 77x^3-71x^2-28x+12$; $30x^3-23x^2+13x-6, 85x^3-86x^2+41x-12$; $x^3-x^2+x-1, 2x^3-4x^2+7x-5$; $x^3-3x^2-3x-4, 2x^3-9x^2+7x-12$.

20. $x^3+2x^2+2x+1, 2x^4+x^3-4x^2-x+2$.

21. $2x^3+5x^2y-9xy^2+2y^3, 2x^4+5x^3y-12x^2y^2+6xy^3-y^4$.

53. To find the G.C.M. of more than two Compound Quantities not easily resolvable. Find (1) the G.C.M. of

the first and second quantities; (2) the G.C.M. of this result and the third; and so on.

Example.—Find the G.C.M. of x^4+x^2-20, x^4-16, and x^2+x-6.

$$\text{G.C.M. of } x^4+x^2-20, \text{ and } x^4-16, = x^2-4;$$
$$\text{G.C.M. of } x^2-4, \text{ and } x^2+x-6, = x-2;$$
$$\therefore \text{ G.C.M. of the three quantities} = x-2, \text{ or } 2-x.$$

Exercise XXV.

1. x^2+5x+6, $x^2+7x+12$, $x^2+8x+15$; x^2+2x-8, x^2-6x+8, $x^2+3x-10$; $x^2-8x+15$, x^2+x-12, x^2-x-6.

2. x^3-7x+6, x^3-4x^2+x+6, x^2-3x+2.

3. $x^3-13x-12$, x^3-4x^2-x+4, x^2-2x-8.

4. x^3-x+6, x^3+x^2-3x+9, x^4+2x^2+9.

5. $2x^3-11x^2+17x-6$, $4x^3-21x+10$, $4x^2+4x-3$.

LEAST COMMON MULTIPLE.

54. One quantity is a *Multiple* of another when that other is one of its factors; a *Common Multiple* of two or more others, when each of them is one of its factors.

Thus, $6a^2x$ is a common multiple of $3a^2$, $2ax$, $6x$, &c.

55. The *Least Common Multiple* of two or more quantities is the quantity of lowest powers that has each of them for one of its factors.

Thus, $12a^2x$ is the lowest common multiple of $3a^2$, $2x^2$, $4ax$, and $6a^2x$. 'Least Common Multiple' (abbreviated L.C.M.) is the name generally used. 'Lowest Common Multiple' would be a better one. In arithmetic, we seek always the *least* C.M.; in algebra, the *lowest*; that is, the one containing lowest powers. (Sect. 52, Note 5.)

56. The L.C.M. of Simple or easily resolvable Compound Quantities may be found almost by inspection by either of the following methods:

1. Resolve them into their elementary factors; the L.C.M. will

contain *all* the factors of the first, those of the second not contained by the first, those of the third not contained by the first, or second; and so on. Thus, since

$$
\begin{array}{lll|lll}
(1.) & 6a^2b^2 = & 2\,.\,3\,.\,aa\ \,.\,bb & (2.) & a^2+ab & = a(a+b) \\
 & 9ab^3 = & 3\,.\,3\,.\,a\ \ \,.\,bbb & & a^2-b^2 & = (a+b)(a-b) \\
 & 12a^3b = & 2\,.\,2\,.\,3\,.\,aaa\,.\,b & & a^2+2ab+b^2 & = (a+b)(a+b)
\end{array}
$$

The L.C.M. of (1) will be $2\,.\,3\,.\,aa\,.\,bb \times 3.b \times 2.a = 36a^3b^3$, $3.b$ being the additional factors of $9ab^3$, and $2.a$ factors of $12a^3b$ additional to those set down for $6a^2b^2$ and $9ab^3$; and the L.C.M. of (2) will be $a(a+b)(a-b)(a+b) = a^4 + a^3b - a^2b^2 - ab^3$.

None but elementary factors should be set down; had we written $12a^3b = 4\,.\,3\,.\,aaa\,.\,bb$, the result for (1) would have been $2\,.\,3\,.\,aa\,.\,bb \times 3.b \times 4.a = 72a^3b^3$.

2. The arithmetical method, virtually the same as 1. Thus,

$$
\begin{array}{r|lll}
2 & 6ab, & 10a^2b, & 15ab^2 \\ \hline
3 & 3ab, & 5a^2b, & 15ab^2 \\ \hline
5 & ab, & 5a^2b, & 5ab^2 \\ \hline
a & ab, & a^2b, & ab^2 \\ \hline
a & b, & ab, & b^2 \\ \hline
b & b, & b, & b^2 \\ \hline
b & 1, & 1, & b \\ \hline
 & 1, & 1, & 1
\end{array}
\qquad
\begin{array}{r|lll}
a & a^2+ab, & a^2-b^2, & a^2+2ab+b^2 \\ \hline
a+b & a+b, & a^2-b^2, & a^2+2ab+b^2 \\ \hline
a-b & 1, & a-b, & a+b \\ \hline
a+b & 1, & 1, & a+b \\ \hline
 & 1, & 1, & 1
\end{array}
$$

The L.C.M. for (2) being
$a(a+b)(a-b)(a+b)$
$= a^4 + a^3b - a^2b^2 - ab^3$, as before.

3. The readiest method for simple quantities is: (1) Find the L.C.M of their coefficients, as in 2; and (2) annex the highest given power of each letter.

57. To find the L.C.M. of Two Compound Quantities not easily resolvable by inspection. (1) Find their G.C.M.; (2) divide one of the quantities by it; and (3) multiply the quotient by the other quantity.

Example.—Find the L.C.M. of $2x^2 - x - 15$ and $3x^2 - 13x + 12$.
Their G.C.M. $= x - 3$; $(2x^2 - x - 15) \div (x - 3) = 2x + 5$;
$(3x^2 - 13x + 12)(2x + 5) = 6x^3 - 11x^2 - 41x + 60$, their L.C.M.

It is better to keep the L.C.M. in factors than to multiply out.

Exercise XXVI.

Find the L.C.M. of

1. x, x^2, x^3; ab, bc, ac; axy, bxz, axz; x^2y, xy^2; a^3b^2, a^2b^3.

2. ax, a^2x, ax^2; $2a, 4b, 8c$; $6ax, 9xz$; $4x^3y^2, 8x^2y^3$.

3. $18ax^2, 12ax^3$; $10x^3y^2, 15x^2z, 6yz^2$; $16ax^2, 24a^2y, 18xy^2$.

4. $ab - ac, b^2 - bc, bc - c^2$; $x^2 - xy, xy - y^2, x^2y - xy^2$.

5. $ax, ax+x^2, a^2+ax, a^2x+ax^2$; $3a, 4x, 2(a+x), 6(a+x)^2$.

6. $a - b, (a - b)^2, (a - b)^3$; $2x - 3x^2, 2y - 3xy, 4xy - 6x^2y$.

7. $a, x, a+x, a-x$; $x+1, 1-x, 1-x^2$; x^2+x, x^2-x, x^3-1.

8. $x^2 + xy, x^2y - y^3$; $a^3 - ab^2, a^2b - b^3$; $5 + x, 25 - x^2$.

9. $2x+1, 2x-1, 4x^2-1$; $2x-4, 2x^2-8$; $6y+2xy, 9x-x^3$.

10. $2a^2 - 8b^2, 2a+4b, 3a-6b$; $x+y, x-y, x^2+y^2, x^2-y^2$; $2x+3, 2x-3, 4x^2-9, 4x^2+9$; $a^2-b^2, a^2+2ab+b^2$; x^2-1, x^2-2x+1; $a^2-2ax+x^2, a^2+2ax+x^2, a^4-2a^2x^2+x^4$.

11. $a+b, a^3+b^3$; $x^3+1, x+1, x^2-x+1$; x^2-y^2, x^3+y^3.

12. x^2-1, x^3-1, x^3+1; $x^3-1, x^2-1, x^2-x, x^2+x+1$: $x^3+1, x^2-1, (x+1)^2, x^2-x+1$; $(1+x)^2, (1-x)^2, 1-x^2, 1+x^2$.

13. x^2+3x+2, x^2+4x+3; $x^2+6x+8, x^2+9x+14$.

14. $x^2-8x+15, x^2-10x+21$; x^2+x-20, x^2-x-30.

15. $4x^2+12x+5, 6x^2-5x-4$; $6x^2+11x-35, 9x^2-3x-20$.

16. $x^3+x^2y+xy^2+y^3, x^3+x^2y+3xy+3y^2$; $a^3-a^2b+ab^2-b^3, a^3-a^2b-ab^2+b^3$; $2x^3+4x^2y-xy^2-2y^3, x^3+2x^2y-2xy^2-4y^3$.

17. x^3-x+6, x^3-5x^2+9x-9; $x^3-2x^2+16, x^3-8x+32$.

58. To find the L.C.M. of more than two such Quantities. Find (1) the L.C.M. of the first and second quantities; (2) the L.C.M. of this result and the third; and so on.

Example.—Find the L.C.M. of $x^4 + x^2 - 20$, $x^4 - 16$, and $x^2 + x - 6$.

L.C.M. of $x^4 + x^2 - 20$ and $x^4 - 16 = x^6 + 5x^4 - 16x^2 - 80$;

L.C.M. of $x^6 + 5x^4 - 16x^2 - 80$ and $x^2 + x - 6$

$$= x^7 + 3x^6 + 5x^5 + 15x^4 - 16x^3 - 48x^2 - 80x - 240,$$

the L.C.M. required.

But to find an L.C.M. of this form requires much labour; and when found, it is generally less convenient to work with than when it is written in factors; thus,

$$x^4 + x^2 - 20 = (x^2 + 5)(x^2 - 4) = (x^2 + 5)(x + 2)(x - 2),$$
$$x^4 - 16 \quad = (x^2 + 4)(x^2 - 4) = (x^2 + 4)(x + 2)(x - 2),$$
$$x^2 + x - 6 = (x + 3)(x - 2);$$
$$\therefore \text{L.C.M.} = (x^2 + 5)(x + 2)(x - 2)(x^2 + 4)(x + 3).$$

EXERCISE XXVII.

1. $x^2 + 8x + 15,\ x^2 + 9x + 18,\ x^2 + 11x + 30.$

2. $x^2 - 9x + 20,\ x^2 - 11x + 28,\ x^2 - 12x + 35.$

3. $x^2 + 2x - 15,\ x^2 - 9x + 18,\ x^2 - x - 30.$

4. $6x^2 + 5x - 6,\ 8x^2 + 2x - 15,\ 12x^2 - 23x + 10;\quad x^2 - 8x + 18,\ x^3 - x^2 + 12,\ x^2 + 5x + 6;\quad x^3 - 15x + 50,\ x^3 - 3x^2 + 20,\ x^2 + 10x + 25.$

FRACTIONS.

59. 1. An *Integer* or *Integral Quantity* is *a whole* or *several wholes* (L. *integer*, whole).

2. A *Fraction* is *a part* or *several parts* of a whole (L. *fractum*, broken).

60. 1. If we wish to have *two-thirds* of an apple, it will come to the same thing whether (1) we take *both the thirds* off *one* apple, or (2) take *a third* off one and *a third* off another, supposing the apples the same in size. Either way we shall have two-thirds, and thus we see that 2 *thirds* of 1 apple = 1 *third* of 2 apples. If we wish to have *three-fourths*, we may take *all the three* off *one* apple, or *a fourth* off one, *a fourth* off another, and *a fourth* off another; and thus we see that 3 *fourths* of 1 = 1 *fourth* of 3. Similarly, 4 *fifths* of 1 = 1 *fifth* of 4; 5 *sevenths* of 1 = 1 *seventh* of 5. Now

1 *third* of 2, $= 2 \div 3$, is written $\frac{2}{3}$; $\therefore$ 2 *thirds* of 1 may be written $\frac{2}{3}$;
1 *fourth* of 3, $= 3 \div 4$, " $\frac{3}{4}$; $\therefore$ 3 *fourths* of 1 " $\frac{3}{4}$;
1 *seventh* of 5, $= 5 \div 7$, " $\frac{5}{7}$; $\therefore$ 5 *sevenths* of 1 " $\frac{5}{7}$.

We now see (1) why a fraction is written like a sum in division; (2) that $\frac{2}{3}$ may be read '2 divided by 3,' '1 third of 2,'

or '2 thirds of 1;' $\frac{3}{4}$, '3 divided by 4,' '1 fourth of 3,' '3 fourths of 1;' &c.; and we also see that

2. (1) The *upper quantity* tells the *number* of parts in the fraction, and is therefore called the *Numerator* (L. *numerus*, number); (2) the *lower* tells the *name*, and therefore the *size*, of the parts, and is consequently called the *Denominator* (L. *nomen*, name).

3. In telling the *size* of the parts, the denominator also tells *how many are required to make a whole;* for, if the parts are *halves*, we know that 2 of them are required; if *thirds*, 3; if *fourths*, 4; and so on.

Therefore (1) the *smaller* the *denominator*, the *fewer* are the *parts* required to make a whole, and therefore the *greater* must be their *size;* (2) the *greater* the *denominator*, the *more numerous* are the *parts* required to make a whole, and therefore the *smaller* must be their *size*. Therefore, *increasing* the *denominator decreases* the *fraction; decreasing* the *denominator increases* the *fraction.*

61. 1. We can *increase* a fraction in two ways; (1) by *increasing* the *number* of its parts, that is, by *increasing* the *numerator;* (2) by *increasing* the *size* of its parts, that is, by *decreasing* the *denominator*. Therefore,

To multiply a fraction, either (1) multiply the numerator, or (2) divide the denominator.

2. We can *decrease* a fraction in two ways; (1) by decreasing the *number* of its parts, that is, by *decreasing* the *numerator;* (2) by decreasing the *size* of its parts, that is, by *increasing* the *denominator*. Therefore,

To divide a fraction, either (1) divide the numerator, or (2) multiply the denominator.

3. If *both* numerator and denominator be *multiplied* by the *same* quantity, the value of the fraction is not altered.

For multiplying the numerator *increases* the *number* of the parts so many times, while multiplying the denominator *decreases* their *size* just as many times.

4. If *both* numerator and denominator be divided by the *same* quantity, the value of the fraction is not altered.

For dividing the numerator *decreases* the *number* of the parts so many times, while dividing the denominator *increases* their *size* just as many times.

5. An integral quantity may be written as a fraction with 1 for its denominator, for dividing it by 1 does not alter its value. Thus, $7 = \frac{7}{1}$, that is, $7 \div 1$.

6. A *negative* fraction becomes *positive*, and a *positive* fraction becomes *negative*, if all the signs of the numerator *or* all those of the denominator be changed. Thus,

$$(1.)\ -\frac{x-y}{x+y} = +\frac{-x+y}{x+y};\ \text{or } (2.)\ -\frac{x-y}{x+y} = +\frac{x-y}{-x-y}.$$

For (1) the value of a quantity is not altered by multiplying it by -1×-1, that is, by 1; nor (2) is it altered by first multiplying it by -1 and then dividing it by -1. And

$$(1.)\ -\frac{x-y}{x+y} \times -1 \times -1 = \frac{x-y}{x+y} \times -1 = \frac{-x+y}{x+y}.$$

$$(2.)\ -\frac{x-y}{x+y} \times -1 \div -1 = \frac{x-y}{x+y} \div -1 = \frac{x-y}{-x-y}.$$

62. 1. A *Mixed Quantity* is one which is partly integral and partly fractional, as $2\frac{2}{3}$, $3 - \frac{1}{4}$, $a + \frac{b}{c}$.

2. An *Improper Fraction* is a fraction merely in form, but an integral or a mixed quantity in reality. All others are *Proper Fractions.*

Therefore, whenever the numerator is *not less*, or contains a power of the leading letter *not lower*, than the denominator, the fraction is *improper;* for then the numerator divided by the denominator will give an integral or a mixed quantity. Thus, $\frac{4}{4}, \frac{5}{4}, \frac{a}{a}, \frac{a^2b}{ab}$, are *improper fractions;* $\frac{3}{4}, \frac{ab}{a^2b}$, *proper fractions;* for

$$\frac{4}{4} = 4 \div 4 = 1;\quad \frac{5}{4} = 1 + \frac{1}{4};\quad \frac{a}{a} = 1;\quad \frac{a^2b}{ab} = a.$$

3. The numerator and denominator are called the *terms* of the fraction; and a fraction is said to be in its *lowest terms* when its numerator and denominator have no common factor.

Exercise XXVIII.

1. Read, in three ways, the fractions $\frac{3}{4}, \frac{5}{7}, \frac{4}{6}, \frac{5}{8}, \frac{8}{9}, \frac{7}{10}, \frac{8}{11}, \frac{11}{12}$.

2. Tell the *number* and *size* of their parts.

3. How many parts are required to make a whole in each?

4. Find the value of $\frac{1}{3} \times 2$, $\frac{6}{7} \div 3$, $\frac{5}{12} \times 7$, $\frac{8}{11} \div 4$, $\frac{3}{17} \times 5$.

5. Find, in two ways, $\frac{5}{6} \cdot 3$, $\frac{9}{10} \div 8$, $\frac{a}{b} \cdot b$, $\frac{ax}{a^2x} \cdot a$, $\frac{8x}{y} \div 4$.

6. Write in fractional form 3, 4, 5, 19, a, ab, $x - y$, $a^2 - b^2$.

7. Write, in two ways, as *positive* fractions, $-\frac{a}{b}$, $-\frac{a-b}{a^2-b^2}$; as *negative* fractions, $\frac{-x}{y}$, $\frac{xy}{-z}$, $\frac{-x+y}{x-y}$, $\frac{x-1}{-x+1}$, $\frac{-4a^3}{x^2-a^2}$.

Reduction of Fractions.

63. To Reduce a Fraction to its Lowest Terms.—Divide both numerator and denominator by their G.C.M.

This makes its form simpler, but does not alter its value. (61. 4).

In practice, the better method is to resolve both numerator and denominator into their factors, and then cast out the factors common to both. Thus,

(1.)
$$\frac{6x^2}{9x^3} = \frac{2 \cdot 3x^2}{3x \cdot 3x^2} = \frac{2}{3x};$$

(2.)
$$\frac{x^3y - xy^3}{x^4 - y^4} = \frac{xy(x^2 - y^2)}{(x^2+y^2)(x^2-y^2)} = \frac{xy}{x^2+y^2};$$

(3.)
$$\frac{x^2 + 5x + 6}{x^2 - x - 12} = \frac{(x+2)(x+3)}{(x-4)(x+3)} = \frac{x+2}{x-4}.$$

Exercise XXIX.

Reduce to their lowest terms:

1. $\frac{ab}{bc}$, $\frac{abc}{bcd}$, $\frac{x^2y}{xy^2}$, $\frac{x^3y^2z}{xy^2z^2}$, $\frac{3a^2b}{6ab^2}$, $\frac{8ac^2}{12a^2c}$, $\frac{9a^3x^2}{21a^2x^3}$, $\frac{a^2 - ax}{a^2 + ax}$.

2. $\frac{xy - y^2}{x^3 - xy^2}$, $\frac{x^2 + x}{x^3 - x^2}$, $\frac{x^2 - x^4}{x^2 - x^6}$, $\frac{a^2b + ab^2}{a^3b - ab^3}$, $\frac{x^3 + xy^2}{x^4y - y^5}$, $\frac{2x + 4}{2x^2 - 8}$.

3. $\frac{1 - 2x}{1 - 4x^2}$, $\frac{3x^2y + xy}{9x^3 - x}$, $\frac{9x + 3x^2}{6x^2 - 54}$, $\frac{a^2 - b^2}{a^3 - b^3}$, $\frac{x^2 - y^2}{x^3 + y^3}$, $\frac{a^3 - b^3}{a^6 - b^6}$.

4. $\frac{x^4 - y^4}{x^6 - y^6}$, $\frac{x^2 - 4}{x^3 - 8}$, $\frac{x - x^3}{1 - x^3}$, $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$, $\frac{x^2 + 2x + 1}{x^3 + 1}$.

5. $\frac{-x-1}{x+1}$, $-\frac{b-a}{a^2-b^2}$, $\frac{x-x^2}{x^2-1}$, $-\frac{x^2-xy}{y^2-x^2}$, $-\frac{a^2-2ab+b^2}{b^2-a^2}$.

6. $\frac{a^3 - 2a^2x + ax^2}{a^3 - ax^2}$, $\frac{3x^2y^2 + 6xy^3 + 3y^4}{9(x^2y^2 - y^4)}$, $\frac{2a^4 - 4a^3b + 2a^2b^2}{6(a^4 - a^2b^2)}$.

7. $\frac{x^2 + 6x + 8}{x^2 + 5x + 6}$, $\frac{x^2 - 8x + 15}{x^2 - 9x + 20}$, $\frac{x^2 + 2x - 15}{x^2 + 3x - 10}$, $\frac{x^2 + x - 6}{x^2 - x - 12}$.

8. $\frac{3x^2 + 6x + 3}{3x^2 + 9x + 6}$, $\frac{2x^2 + 6x - 8}{2x^2 + 8x - 10}$, $\frac{6x^2 + x - 12}{8x^2 + 6x - 9}$, $\frac{x^2 + x - 2}{x^3 + 2x^2 - x - 2}$.

64. To Reduce an Improper Fraction to an Integral or a Mixed Quantity.—Divide its numerator by its denominator. Thus,

$$\frac{8x^2y^3}{4xy^2} = 2xy; \quad \frac{7x}{3} = 2x + \frac{x}{3}; \quad \frac{x^3 - 2x^2 - 1}{x^2 - x} = x - 1 - \frac{x + 1}{x^2 - x}.$$

This is merely performing the division indicated, for these fractions mean $8x^2y^3 \div 4xy^2$, $7x \div 3$, &c. In the third example, the remainder is $-x - 1$, but the line between numerator and denominator is really a bracket, and $-x - 1 = -(x + 1)$.

Exercise XXX.

Reduce to integral or mixed quantities:

1. $\frac{9}{3}$, $\frac{15}{4}$, $\frac{6a}{3a}$, $\frac{5x}{2}$, $\frac{x + a}{x}$, $\frac{x^3 - a^2}{x^2}$, $\frac{4x^4 + 3y}{2x^2}$, $\frac{x^2 + x - 2}{x}$.

2. $\frac{6x^3 - 2ax}{3x^2}$, $\frac{x^2 - 2xy + y^2}{x - y}$, $\frac{a^2 + 2ab - b^2}{a + b}$, $\frac{a^2 + b^2}{a - b}$, $\frac{x^4 - 1}{x + 1}$.

3. $\frac{x^4 + 1}{x + 1}$, $\frac{x^3 - 1}{x^2 + 1}$, $\frac{x^3 - x^2 + 1}{x^2 + 1}$, $\frac{x^3 - 2x^2 + 1}{x^2 - x + 1}$, $\frac{x^3 - 2x}{x^2 + x + 1}$.

65. To Reduce a Mixed Quantity to an Improper Fraction.—(1) Multiply the integral part by the denominator of the fractional part; (2) add the numerator of the latter to the product; and (3) take the result as a numerator, and the denominator of the fractional part as a denominator. Thus,

$$2x + \frac{x}{3} = \frac{6x + x}{3} = \frac{7x}{3}; \quad x - 1 - \frac{x - 1}{x + 1} = \frac{x^2 - 1 - (x - 1)}{x + 1} = \frac{x^2 - x}{x + 1}.$$

This is properly addition; (1) the integral part being written as a fraction with 1 for its denominator (61. 5), (2) this fraction being changed to another having the same denominator as the fractional part (66. 1), and (3) their numerators being then collected. Thus,

$$2x+\frac{x}{3}=\frac{2x}{1}+\frac{x}{3}=\frac{6x}{3}+\frac{x}{3}=\frac{6x+x}{3}=\frac{7x}{3}.$$

$$x-1-\frac{x-1}{x+1}=\frac{x-1}{1}-\frac{x-1}{x+1}=\frac{x^2-1}{x+1}-\frac{x-1}{x+1}=\frac{x^2-1-(x-1)}{x+1}.$$

Note that the $-$ before the fraction, when removed to the numerator, changes all the signs of the latter (61. 6), so that

$$\frac{x^2-1}{x+1}-\frac{x-1}{x+1}=\frac{x^2-1}{x+1}+\frac{-(x-1)}{x+1}=\frac{x^2-1-(x-1)}{x+1}.$$

Exercise XXXI.

Reduce to improper fractions:

1. $2+\frac{1}{3}$, $4-\frac{2}{5}$, $a+\frac{c}{b}$, $x-\frac{y}{z}$, $1+\frac{a^2}{ax}$, $a^2+\frac{b^2}{ab}$, $a+\frac{b-c}{d}$.

2. $x-\frac{x+1}{x}$, $a+\frac{ab}{a-b}$, $x-\frac{xy}{x+y}$, $a-\frac{ax}{a-x}$, $x+y-\frac{4xy}{x+y}$.

3. $a-b+\frac{4ab}{a-b}$, $a+x-\frac{2ax}{a-x}$, $a+\frac{b^2-ab}{a-b}$, $a-\frac{ab+b^2}{a+b}$.

4. $x-y-\frac{x^2+y^2}{x-y}$, $a-x-\frac{x^2-a^2}{a+x}$, $x^2+y^2-\frac{xy(y-x)}{x-y}$.

66. 1. To Reduce a Fraction to an Equivalent one having a Given Denominator.—(1) Divide the given denominator by the denominator of the fraction, and (2) multiply both the numerator and denominator of the fraction by the quotient (61. 3).

Examples.

Change (1) $\frac{2x^2}{3xy^2}$ to an equivalent fraction having $12x^2y^2$ for its denominator, and (2) $\frac{xy}{x^2+y^2}$ to an equivalent one having x^4-y^4 for its denominator.

Since $12x^2y^2 \div 3xy^2 = 4x$; and $x^4-y^4 \div x^2+y^2 = x^2-y^2$;

$$\therefore\ (1.)\quad \frac{2x^2}{3xy^2}=\frac{2x^2\times 4x}{3xy^2\times 4x}=\frac{8x^3}{12x^2y^2}.$$

$$(2.)\quad \frac{xy}{x^2+y^2}=\frac{xy(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}=\frac{x^3y-xy^3}{x^4-y^4}.$$

2. TO REDUCE FRACTIONS TO EQUIVALENT ONES HAVING THE LEAST COMMON DENOMINATOR.—Find the L.C.M. of their denominators (56, &c.), and reduce each fraction to an equivalent one having this L.C.M. for its denominator (66. 1).

Examples.

1. Change (1) $\frac{x}{yz}$, $\frac{y}{xz}$; (2) $\frac{a}{a+b}$, $\frac{b}{a-b}$; (3) $\frac{1}{x^2+7x+12}$, $\frac{1}{x^2-x-12}$, to equivalent fractions having the L.C.D.

The L.C.M. of the denominator in (1) is xyz; in (2), $(a+b)(a-b)$; in (3), $(x+3)(x+4)(x-4)$. Therefore,

(1.) $\frac{x}{yz}=\frac{x^2}{xyz}$ (2). $\frac{a}{a+b}=\frac{a(a-b)}{(a+b)(a-b)}=\frac{a^2-ab}{(a+b)(a-b)}$

$\frac{y}{xz}=\frac{y^2}{xyz}$. $\frac{b}{a-b}=\frac{b(a+b)}{(a-b)(a+b)}=\frac{ab+b^2}{(a+b)(a-b)}$.

(3.) $\frac{1}{x^2+7x+12}=\frac{1}{(x+3)(x+4)}=\frac{x-4}{(x+3)(x+4)(x-4)}$.

$\frac{1}{x^2-x-12}=\frac{1}{(x+3)(x-4)}=\frac{(x+4)}{(x+3)(x+4)(x-4)}$.

2. Change (1) $\frac{x}{x+y}$, $\frac{y}{y-x}$, $\frac{xy}{x^2-y^2}$; (2) $\frac{1}{x-1}$, $\frac{1}{1-x}$, $\frac{x^2}{1-x^2}$, to equivalent fractions having the L.C.D.

Where a denominator of the form $y-x$ occurs with others of the form $x-y$, $x+y$, x^2-y^2, &c., it is better, that they may all be alike in form, to multiply the numerator and denominator of one or more of the fractions by -1. Thus,

(1.) $\frac{x}{x+y}=\frac{x}{x+y}$ (2.) $\frac{1}{x-1}=\frac{-1}{1-x}$ or $\frac{1}{x-1}$

$\frac{y}{y-x}=\frac{-y}{x-y}$ $\frac{1}{1-x}=\frac{1}{1-x}$ or $\frac{-1}{x-1}$

$\frac{xy}{x^2-y^2}=\frac{xy}{x^2-y^2}$. $\frac{x^2}{1-x^2}=\frac{x^2}{1-x^2}$ or $\frac{-x^2}{x^2-1}$.

This does not alter their value (61. 3), and the L.C.D. is then more easily found.

Exercise XXXII.

1. Reduce ax, $3x$, $\frac{2a}{x}$, $\frac{x^2}{3ax}$, a^2x, $-x^3$, $-\frac{3a^2}{4ax}$, $-\frac{5ax^2}{12a^2x}$, to equivalent fractions having $12a^2x^2$ for their denominators.

Reduce to equivalent fractions having the L.C.D:

2. $\frac{3}{x}, \frac{5}{y}$; $\frac{a}{x}, \frac{b}{x^2}, \frac{c}{x^3}$; $\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}$; $\frac{2}{ax}, \frac{3}{a^2x}, \frac{4a}{a^2x^2}$.

3. $\frac{1}{a+b}, \frac{a-b}{(a+b)^2}$; $\frac{1}{a-b}, \frac{a+b}{(a-b)^2}, \frac{4ab}{(a-b)^3}$; $\frac{1}{x+2}, \frac{1}{x+3}$.

4. $\frac{a}{a+x}, \frac{a^2}{a^2+ax}$; $\frac{x}{x-y}, \frac{y^2}{x^2-xy}$; $\frac{1}{ab+b^2}, \frac{1}{a^2+ab}, \frac{1}{a^2b+ab^2}$.

5. $\frac{a}{a+x}, \frac{x}{a-x}$; $\frac{1}{x-y}, \frac{1}{x+y}, \frac{2y}{x^2-y^2}$; $\frac{x}{x+1}, \frac{x}{x-1}, \frac{x^2}{x^2-1}$.

6. $\frac{3x}{x+2}, \frac{4}{x^2-4}$; $\frac{2x}{3x-9}, \frac{5}{2x^2-18}$; $\frac{3x}{1+3x}, \frac{5x}{2-6x}, \frac{6}{1-9x^2}$.

7. $\frac{a}{a+b}, \frac{b}{b-a}, \frac{2ab}{a^2-b^2}$; $\frac{a}{a-x}, \frac{x}{x-a}, \frac{a^2}{a^2-x^2}, \frac{x^2}{x^2-a^2}$.

8. $\frac{a+x}{a-x}, \frac{a-x}{a+x}$; $\frac{a}{a+b}, \frac{b}{a-b}, \frac{ab}{a^2+b^2}$; $\frac{1}{x-1}, \frac{x}{x^2-1}, \frac{x}{x^3+1}$.

9. $\frac{2}{(x+3)(x+4)}, \frac{3}{(x+4)(x+5)}$; $\frac{1}{(a-b)(a-c)}, \frac{1}{(a-b)(b-c)}$.

10. $\frac{1}{(a-b)(a-c)}, \frac{1}{(a-b)(c-a)}$; $\frac{x}{(x-y)(z-x)}, \frac{y}{(y-x)(x-z)}$.

11. $\frac{1}{x-y}, \frac{xy}{x^3-y^3}$; $\frac{1}{x+2}, \frac{3x^2-6x}{x^3+8}$; $\frac{a+x}{a^2-ax+x^2}, \frac{a^2-x^2}{a^3+x^3}$.

12. $\frac{x-2}{x^2+5x+6}, \frac{x-4}{x^2+6x+8}$; $\frac{x-2}{x^2+6x+8}, \frac{x-6}{x^2+2x-8}, \frac{x+4}{x^2-4}$.

13. $\frac{x+3}{2x^2+x-1}, \frac{x-3}{2x^2-3x+1}$; $\frac{4x+5}{6x^2+13x+6}, \frac{3x-2}{8x^2+2x-15}$.

Addition and Subtraction.

67. Rule.—(1) Change the fractions to equivalent ones having the L.C.D. (66. 2); (2) collect their numerators for the numerator of the answer, and keep the L.C.D. for its denominator.

Before fractions can be added or subtracted, their *parts* must be of *the same sort* (sect. 21), and therefore of *the same size*; that is, they must have a C.D. Then, to find the *number* of *parts* in all of them, we collect their numerators (60. 2); and keep the C.D. to tell their *size*.

Thus, just as 2 *apples* + 3 *apples* = 5 *apples*, so 2 *sevenths* + 3 *sevenths* = 5 *sevenths*; that is, $\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$. But we cannot add 2 *fifths* and 3 *fourths* till they have a C.D.

Examples.

1. $\frac{x-y}{x+y}+\frac{x+y}{y-x}+\frac{4y^2}{x^2-y^2}=\frac{x-y}{x+y}-\frac{x+y}{x-y}+\frac{4y^2}{x^2-y^2}$

$$=\frac{(x-y)(x-y)}{(x+y)(x-y)}-\frac{(x+y)(x+y)}{(x-y)(x+y)}+\frac{4y^2}{(x+y)(x-y)}$$

$$=\frac{x^2-2xy+y^2-(x^2+2xy+y^2)+4y^2}{(x+y)(x-y)}=\frac{-4xy+4y^2}{(x+y)(x-y)}$$

$$=-\frac{4xy-4y^2}{(x+y)(x-y)}=-\frac{4y(x-y)}{(x+y)(x-y)}=-\frac{4y}{x+y}.$$

2. $\frac{x^2}{(x-y)(x-z)}-\frac{y^2}{(y-x)(z-x)}$

$$=\frac{x^2}{(x-y)(x-z)}-\frac{y^2}{(x-y)(x-z)}=\frac{x^2-y^2}{(x-y)(x-z)}=\frac{x+y}{x-z}.$$

3. $\frac{x-6}{x^2+14x+48}+\frac{x-6}{x^2+10x+24}=\frac{x-6}{(x+8)(x+6)}+\frac{x-6}{(x+6)(x+4)}$

$$=\frac{(x-6)(x+4)}{(x+8)(x+6)(x+4)}+\frac{(x-6)(x+8)}{(x+8)(x+6)(x+4)}$$

$$=\frac{x^2-2x-24+x^2+2x-48}{(x+8)(x+6)(x+4)}$$

$$=\frac{2x^2-72}{(x+8)(x+6)(x+4)}=\frac{2(x^2-36)}{(x+8)(x+6)(x+4)}=\frac{2(x-6)}{(x+8)(x+4)}.$$

In 1, $+\frac{x+y}{y-x}$ is changed to $-\frac{x+y}{x-y}$, and $\frac{-4xy+4y^2}{(x+y)(x-y)}$ to

$-\frac{4xy-4y^2}{(x+y)(x-y)}$ (61. 6; 66. 2, Note).

In 2, $-\frac{y^2}{(y-x)(z-x)}$ is changed first to $-\frac{-y^2}{(x-y)(z-x)}$, and then to $-\frac{y^2}{(x-y)(x-z)}$, by multiplying its numerator and denominator by -1, and then by -1 again. We multiply a quantity when we multiply *one* of its factors; therefore, multiplying $(y-x)(z-x)$ by -1, changes the signs of $y-x$; multiplying again by -1, changes those of $z-x$.

In adding several fractions, it is often better to add first two of them, then a third to their sum, and so on. Thus,

4. $\frac{1}{a-b}+\frac{1}{a+b}-\frac{2a}{a^2+b^2}=\frac{4ab^2}{a^4-b^4}$. For

(1.) $\frac{1}{a-b}+\frac{1}{a+b}=\frac{a+b}{a^2-b^2}+\frac{a-b}{a^2-b^2}=\frac{a+b+a-b}{a^2-b^2}=\frac{2a}{a^2-b^2}$.

(2.) $\frac{2a}{a^2-b^2}-\frac{2a}{a^2+b^2}=\frac{2a(a^2+b^2)}{a^4-b^4}-\frac{2a(a^2-b^2)}{a^4-b^4}$

$$=\frac{2a^3+2ab^2-(2a^3-2ab^2)}{a^4-b^4}=\frac{4ab^2}{a^4-b^4}.$$

Exercise XXXIII.

1. $\frac{a}{4}+\frac{a}{6}$; $\frac{2a}{3}+\frac{4a}{5}$; $\frac{7x}{5}-\frac{5x}{7}$; $\frac{x+2y}{4}+\frac{2x-y}{3}-\frac{5x+y}{6}$.
2. $\frac{a-b}{a}+\frac{a+b}{b}$; $\frac{x+y}{y}-\frac{x-y}{x}$; $\frac{m+n}{m}+\frac{m-n}{n}-\frac{m^2-n^2}{mn}$.
3. $\frac{x-y}{xy}+\frac{y-z}{yz}+\frac{z-x}{zx}$; $\frac{x^2-yz}{yz}+\frac{y^2-zx}{zx}+\frac{z^2-xy}{xy}$.
4. $\frac{m-3n}{3n}-\frac{m-6n}{6n}+\frac{m-9n}{9n}$; $\frac{x+3}{13x}+\frac{x^2+5}{26x^2}-\frac{x^3+7}{39x^3}$.
5. $\frac{2}{x+2}+\frac{2}{x+4}$; $\frac{a}{a+x}+\frac{a}{a-x}$; $\frac{1}{1-x}+\frac{1}{1+x}$.
6. $\frac{3}{x+4}-\frac{3}{x+1}$; $\frac{a}{a+x}-\frac{a}{a-x}$; $\frac{1}{1+m}-\frac{1}{1-m}$.
7. $\frac{1}{x+1}-\frac{1}{x-1}$; $\frac{x}{x+y}-\frac{y}{x-y}$; $\frac{a}{a+b}+\frac{ab}{a^2+ab}$.
8. $\frac{a}{(a+x)^2}-\frac{2ax}{(a+x)^3}$; $\frac{1}{a-x}+\frac{a+x}{(a-x)^2}-\frac{2(ax-x^2)}{(a-x)^3}$.

9. $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$; $\dfrac{2x+1}{2x-1}-\dfrac{2x-1}{2x+1}$; $\dfrac{x}{xy-y^2}-\dfrac{2x-y}{x^2-xy}$.

10. $\dfrac{a}{a+b}+\dfrac{b}{a-b}+\dfrac{2ab}{a^2-b^2}$; $\dfrac{x}{x^2-xy}+\dfrac{2x}{x^2-y^2}-\dfrac{xy}{x^2y+xy^2}$.

11. $\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}-\dfrac{(a+b)^2}{a^2-b^2}$; $\dfrac{x}{x-1}-\dfrac{x}{x+1}-\dfrac{2x^2}{x^2-1}$.

12. $\dfrac{3}{a-2b}+\dfrac{4}{2a+4b}-\dfrac{5}{a^2-4b^2}$; $\dfrac{2}{1-3a}-\dfrac{3a}{1+3a}+\dfrac{4-5a}{1-9a^2}$.

13. $\dfrac{a}{a^3+b^3}+\dfrac{a}{a^3-b^3}-\dfrac{2a^2b^3}{a^6-b^6}$; $\dfrac{4+3a}{3-a}-\dfrac{2+4a}{3+a}-\dfrac{3+2a+7a^2}{9-a^2}$.

14. $\dfrac{1}{2a-2}+\dfrac{1}{3a+3}-\dfrac{5a}{6a^2-6}$; $\dfrac{xy}{x^2-y^2}+\dfrac{xy}{x^2+y^2}-\dfrac{2xy^3}{x^4-y^4}$.

15. $\dfrac{1}{x+2}-\dfrac{3}{2x-4}-\dfrac{x-10}{3x^2-12}$; $\dfrac{a^2+x^2}{ax}-\dfrac{a^2}{ax+x^2}-\dfrac{x^2}{a^2+ax}$.

16. $\dfrac{2}{x}+\dfrac{5}{1-2x}-\dfrac{2x+9}{1-4x^2}$; $\dfrac{3x-a}{a-x}+\dfrac{x-3a}{a+x}+\dfrac{4a(a-x)}{a^2-x^2}$.

17. $\dfrac{x-4}{x^2-15x+54}+\dfrac{x+4}{x^2+3x-54}$; $\dfrac{x-3}{x^2-x-12}+\dfrac{x+3}{x^2-7x+12}$.

18. $\dfrac{x-4}{x^2+14x+40}+\dfrac{x-4}{x^2+2x-8}$; $\dfrac{x+2}{x^2+7x+12}-\dfrac{x+1}{x^2+8x+15}$.

19. $\dfrac{a-7}{a^2-8a+15}-\dfrac{a-3}{a^2-12a+35}$; $\dfrac{a-9b}{a^2-6ab+5b^2}-\dfrac{a-b}{a^2-14ab+45b^2}$.

20. $\dfrac{x+3}{x^2-x-2}+\dfrac{x+1}{x^2+x-6}-\dfrac{x+2}{x^2+4x+3}$.

21. $\dfrac{x}{x-y}-\dfrac{y}{y-x}$; $\dfrac{x}{x+y}-\dfrac{xy}{y^2-x^2}$; $\dfrac{x}{x+1}+\dfrac{x}{1-x}+\dfrac{2x^4}{x^4-x^2}$.

22. $x-\dfrac{x}{1-x}-\dfrac{x^2}{1+x}$; $4a+\dfrac{x^2-ax}{x+a}-\dfrac{x^2+ax}{x-a}$; $\dfrac{a}{a-b}+\dfrac{b}{b-a}$.

23. $\dfrac{a}{a-x}+\dfrac{x}{x-a}-\dfrac{2(ax-x^2)}{a^2-x^2}$; $\dfrac{x}{(x-y)(x-z)}+\dfrac{y}{(x-y)(z-x)}$.

24. $\dfrac{a^2}{(a-b)(a-c)}-\dfrac{c^2}{(b-a)(c-a)}$; $\dfrac{c}{(a-b)(c-a)}-\dfrac{a}{(b-a)(a-c)}$.

25. $\dfrac{1}{(a-x)(b+x)}+\dfrac{1}{(x-a)(a+b)}-\dfrac{1}{(a+b)(x+b)}$.

26. $\dfrac{1}{x-2}-\dfrac{x}{x^2-4}+\dfrac{x+2}{x^3-8}$; $\dfrac{2}{a+b}+\dfrac{a+b}{a^2-ab+b^2}-\dfrac{2(a^2+b^2)}{a^3+b^3}$.

27. $\dfrac{2}{x+1}-\dfrac{x+4}{x^2-x+1}+\dfrac{6x+3}{x^3+1}$; $\dfrac{1}{a-3}-\dfrac{a+3}{a^2+3a+9}+\dfrac{6(a+1)}{a^3-27}$.

28. $\frac{a+b}{b}-\frac{2a}{a+b}-\frac{a^3-a^2b}{a^2b-b^3}$; $\frac{1}{a-b}-\frac{1}{a+b}-\frac{2b}{a^2+b^2}$.

29. $\frac{x}{a-x}-\frac{x^2}{a^2-x^2}+\frac{ax}{a^2+x^2}$; $\frac{xy}{xy+y^2}+\frac{xy}{x^2-xy}-\frac{x^2-y^2}{x^2+y^2}$.

30. $\frac{x}{x-y}-\frac{x}{x+y}-\frac{2xy}{x^2+y^2}$; $\frac{1}{a-x}-\frac{1}{a+x}-\frac{3x}{a^2-x^2}+\frac{a^3+ax^2}{a^4-x^4}$.

31. $\frac{1}{x-y}-\frac{1}{x+y}-\frac{2y}{x^2+y^2}-\frac{4y^3}{x^4+y^4}$; $\frac{a}{5-x}-\frac{a}{5+x}-\frac{2ax}{25+x^2}$.

32. $\frac{xy}{3+4z}+\frac{xy}{3-4z}-\frac{6xy}{9+16z^2}$; $\frac{1}{3-a^2}-\frac{1}{3+a^2}+\frac{2a^2}{9+a^4}$.

33. $\frac{x+1}{x-2}-\frac{x+2}{x-1}-\frac{3}{x^2-4}$; $\frac{1}{4-4a}-\frac{1}{4+4a}+\frac{a}{2+2a^2}-\frac{a}{1+a^4}$.

34. $\frac{x-2y}{x+3y}-\frac{x-3y}{x+2y}-\frac{5y^2}{x^2-9y^2}$; $\frac{1}{xy-3}-\frac{1}{xy+3}-\frac{6}{x^2y^2+9}$.

35. $\frac{3x+3y}{x^2+xy+y^2}-\frac{3x-3y}{x^2-xy+y^2}+\frac{2y^3}{x^4+x^2y^2+y^4}$.

36. $\frac{2}{(a+1)(a+2)}+\frac{3}{(a+2)(a+3)}-\frac{4}{(a+3)(a+1)}$.

37. $\frac{bc}{(a-b)(a-c)}+\frac{ac}{(b-a)(b-c)}+\frac{ab}{(c-a)(c-b)}$.

38. $\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}+\frac{(b-c)(c-a)(a-b)}{abc}$.

39. $\frac{(a-c)^2-b^2}{a^2-(b+c)^2}+\frac{(b-a)^2-c^2}{b^2-(c+a)^2}+\frac{(c-b)^2-a^2}{c^2-(a+b)^2}$.

40. $\frac{1}{16(1+x)}+\frac{1}{16(1-x)}+\frac{1}{8(1+x^2)}+\frac{1}{4(1+x^4)}+\frac{1}{2(1+x^8)}$.

MULTIPLICATION AND DIVISION.

68. 1. TO MULTIPLY FRACTIONS TOGETHER.—Multiply together their numerators for the numerator of the product, and their denominators for its denominator.

$\frac{3}{4}$ is the *fourth* part of 3 (60. 1); that is, $\frac{3}{4}$ is 4 times less than 3. Therefore, if a quantity be multiplied by $\frac{3}{4}$, the product will be 4 times less than if it were multiplied by 3.

$$\therefore \tfrac{2}{5} \times \tfrac{3}{4} \text{ is 4 times less than } \tfrac{2}{5} \times 3. \quad \text{But } \tfrac{2}{5} \times 3 = \tfrac{6}{5};$$
$$\therefore \tfrac{2}{5} \times \tfrac{3}{4} = \tfrac{6}{5} \div 4 = \tfrac{6}{20}; \quad \therefore \tfrac{2}{5} \times \tfrac{3}{4} = \tfrac{2 \times 3}{5 \times 4}.$$

Similarly, $\frac{c}{d}$ is the dth part of c; that is, $\frac{c}{d}$ is d times less than c;

$$\therefore \frac{a}{b} \times \frac{c}{d} \text{ is } d \text{ times less than } \frac{a}{b} \times c. \quad \text{But } \frac{a}{b} \times c = \frac{ac}{b};$$

$$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{ac}{b} \div d = \frac{ac}{bd}; \quad \therefore \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

2. To Divide one Fraction by another.—Invert the divisor, and it becomes a multiplier.

Since the *greater* the *divisor* is, the *less* is the *quotient;* and the *less* the *divisor*, the *greater* the *quotient*. Therefore,

$$\tfrac{2}{5} \div \tfrac{3}{4} \text{ is 4 times } \textit{greater} \text{ than } \tfrac{2}{5} \div 3. \quad \text{But } \tfrac{2}{5} \div 3 = \tfrac{2}{15};$$

$$\therefore \tfrac{2}{5} \div \tfrac{3}{4} = \tfrac{2}{15} \times 4 = \tfrac{8}{15}; \text{ so that, } \tfrac{2}{5} \div \tfrac{3}{4} = \tfrac{2}{5} \times \tfrac{4}{3}.$$

Similarly, $\frac{a}{b} \div \frac{c}{d}$ is d times *greater* than $\frac{a}{b} \div c$. But $\frac{a}{b} \div c = \frac{a}{bc}$;

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{a}{bc} \times d = \frac{ad}{bc}; \text{ so that } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Examples.

1. $\frac{2a}{3b} \times \frac{6b}{8a} = \frac{2a \times 6b}{3b \times 8a} = \frac{12ab}{24ab} = \frac{1}{2}.$

2. $\frac{2a}{3b} \div \frac{6b}{8a} = \frac{2a}{3b} \times \frac{8a}{6b} = \frac{2a \cdot 8a}{3b \cdot 6b} = \frac{16a^2}{18b^2} = \frac{8a^2}{9b^2}.$

3. $\frac{a^2 - ab}{a^2 + ab} \times \frac{ab + b^2}{a^2 - b^2} = \frac{a(a - b) \times b(a + b)}{a(a + b) \times (a + b)(a - b)} = \frac{b}{a + b}.$

4. $\frac{x^2 + xy}{x^2y - xy^2} \div \frac{xy^2 + y^3}{x^2y - y^3} = \frac{x(x + y) \times y(x + y)(x - y)}{xy(x - y) \times y^2(x + y)} = \frac{x + y}{y^2}.$

5. $(x^2 - x) \times \frac{x+1}{x^2} \div \frac{x^4 - 1}{x^3 + x} = \frac{x(x-1)}{1} \times \frac{x+1}{x^2} \times \frac{x(x^2+1)}{(x^4 - 1)} = 1.$

6. $\frac{x^2+4x+3}{x^2+5x+4} \cdot \frac{x^2+2x-8}{x^2 + x - 6} = \frac{(x+3)(x+1) \cdot (x+4)(x-2)}{(x+4)(x+1) \cdot (x+3)(x-2)} = 1.$

Exercise XXXIV.

1. $\frac{2x}{9y^3} \times 3y$; $\quad \frac{6x^2}{7y} \div 3x$; $\quad \frac{3ab}{35xy^2} \cdot 5xy$; $\quad \frac{36x^3y^2}{5ab} \div 9x^2y^2$.

2. $\dfrac{15y^2}{16x^2}\cdot\dfrac{4x}{5y}$; $\dfrac{8a}{4x}\div\dfrac{9a}{8x}$; $\dfrac{5a^2b}{6cd^2}\cdot\dfrac{3c^2d}{4ab^2}$; $\dfrac{4a^2c}{5bd^2}\div\dfrac{8ac^2}{10b^2d}$.

3. $\dfrac{a-b}{a+b}\times\dfrac{(a+b)^2}{a^2-b^2}$; $\dfrac{x+y}{x^2-xy}\div\dfrac{x^2-y^2}{(x-y)^2}$; $\dfrac{a^2-ab}{ab+b^2}\cdot\dfrac{a^2+ab}{ab-b^2}$.

4. $\dfrac{x^2+xy}{x^2-y^2}\cdot\dfrac{xy-y^2}{x+y}$; $\dfrac{a^3+a^2b}{a^3-ab^2}\div\dfrac{a^2+b^2}{ab-b^2}$; $\dfrac{x+x^2}{1-x^2}\cdot\dfrac{1-x}{x+x^2}$.

5. $\dfrac{x^2-x}{x^2+x}\div\dfrac{x-x^2}{x+x^2}$; $\dfrac{x^2+x}{x^2-1}\div\dfrac{x+1}{x-x^2}$; $\dfrac{a^2-ab}{a^2+2ab+b^2}\cdot\dfrac{a^2b-b^3}{a^2-2ab+b^2}$.

6. $\dfrac{x^3-x^2}{1+2x+x^2}\div\dfrac{x^2-2x+1}{x^2+x^3}$; $\dfrac{a^2-ax}{x}\cdot\dfrac{ax+x^2}{ax}\div\dfrac{a^4-x^4}{a^3+ax^2}$.

7. $\dfrac{x^2-2x}{x^2+3x}\div\dfrac{3x^2-6x}{2x^2+6x}$; $\dfrac{x^2-3x}{3x^2-12x}\cdot\dfrac{x^3-16x}{2x^2-18}\div\dfrac{2x^2+8x}{3x+9}$.

8. $\dfrac{3x-x^2}{2+2x}\div\dfrac{36-4x^2}{3-3x^2}\cdot\dfrac{6+2x}{y-xy}$; $\dfrac{x^2+5x+6}{x^2+6x+8}\cdot\dfrac{x^2+9x+20}{x^2+9x+18}$.

9. $\dfrac{x^2y+xy}{x^5-x}\cdot\dfrac{x^4-x^2}{xy-y}\div\dfrac{x^3y-xy}{x^2y^2+y^2}$; $\dfrac{x^2-4}{x^2-x-12}\div\dfrac{x^2+3x-10}{x^2+8x+15}$.

10. $\dfrac{a^2-b^2}{a^3+b^3}\cdot\dfrac{a^2-ab+b^2}{a^2-2ab+b^2}$; $\dfrac{a^3-x^3}{a-x}\cdot\dfrac{a^2+ax}{a^2+ax+x^2}\div\dfrac{a^4-x^4}{ax-x^2}$.

11. $\dfrac{x^4-y^4}{x^3-y^3}\cdot\dfrac{x^2+xy+y^2}{x^2+2xy+y^2}\div\dfrac{x^3y+xy^3}{x^2-y^2}$; $\left(a+\dfrac{x^2}{a}\right)\left(a-\dfrac{x^2}{a}\right)$.

12. $\dfrac{x^2-3x+2}{x^4-3x^2-4}\cdot\dfrac{x^4+2x^2+1}{x^2-x+1}\div\dfrac{x^4-1}{x^4+x}$; $\left(\dfrac{a}{x}-\dfrac{x}{a}\right)\cdot\dfrac{ax}{a+x}$.

13. $\dfrac{x^2+8x+16}{x^2+7x+12}\div\dfrac{x^2-2x+4}{x^2+x-6}\div\dfrac{x^2-4}{x^3+8}$; $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\div\left(\dfrac{1}{b}+\dfrac{1}{a}\right)$.

14. $\dfrac{a+b}{a-b}\cdot\dfrac{b-a}{a^2-b^2}$; $\dfrac{a-x}{a^3-x^3}\div\dfrac{x-a}{x+a}$; $\dfrac{x+y}{y}\cdot\dfrac{y-x}{x}\cdot\dfrac{x^2+y^2}{x^4-y^4}$.

15. $\dfrac{y+x}{x-y}\div\dfrac{x+y}{y-x}\cdot\dfrac{y-x}{x^2-y^2}$; $\left(a-\dfrac{ax}{a+x}\right)\left(x+\dfrac{ax}{x-a}\right)$.

16. $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)\cdot\dfrac{xy}{x-y}\div\left(\dfrac{1}{y}+\dfrac{1}{x}\right)$; $\left(\dfrac{1}{y^2}-\dfrac{1}{x^2}\right)\div\left(2+\dfrac{x}{y}+\dfrac{y}{x}\right)\cdot\dfrac{xy}{x-y}$.

FRACTIONAL SIMPLE EQUATIONS.

69. (a) Rule.—Clear the equation of fractions by multiplying every term by the L.C.M. of their denominators, and apply the rule already given in sect. 49 (a).

1. Solve $\frac{2x}{3} - \frac{x}{4} = \frac{x}{6} + 3$.

Multiplying by 12, the L.C.D.,

$$\frac{24x}{3} - \frac{12x}{4} = \frac{12x}{6} + 36$$

that is, $8x - 3x = 2x + 36$

Transposing, $8x - 3x - 2x = 36$

Collecting, $3x = 36$

Dividing by 3, $x = 12$

2. Solve $x - \frac{x+5}{6} = \frac{7x-4}{9}$.

Multiplying by 18, the L.C.D.,

$$18x - \frac{18(x+5)}{6} = \frac{18(7x-4)}{9}$$

$\therefore 18x - 3(x+5) = 2(7x-4)$

$\therefore 18x - 3x - 15 = 14x - 8$

$\therefore 18x - 3x - 14x = -8 + 15$

$\therefore x = 7$

Note 1.—We may shorten the working of examples like the above by mentally multiplying each numerator by the quotient obtained by dividing the L.C.D. by its denominator.

Note 2.—Keep the numerator in a bracket after removing the denominator, as in 2, especially if the fraction be negative.

3. Solve

$$\frac{3x-5}{6} - \frac{3x-1}{7x-1} = \frac{2x-5}{4}$$

$$\frac{3x-5}{6} - \frac{2x-5}{4} = \frac{3x-1}{7x-1}$$

$$\frac{2(3x-5) - 3(2x-5)}{12} = \frac{3x-1}{7x-1}$$

$$\frac{5}{12} = \frac{3x-1}{7x-1}$$

$$5(7x-1) = 12(3x-1)$$

whence $x = 7$

4. Solve

$$\frac{x-3}{x-4} - \frac{x-2}{x-3} = \frac{x-6}{x-7} - \frac{x-5}{x-6}$$

$$\therefore \frac{x^2-6x+9-(x^2-6x+8)}{(x-4)(x-3)}$$

$$= \frac{x^2-12x+36-(x^2-12x+35)}{(x-7)(x-6)}$$

$$\therefore \frac{1}{(x-4)(x-3)} = \frac{1}{(x-7)(x-6)}$$

$\therefore x^2 - 13x + 42 = x^2 - 7x + 12$

whence $x = 5$

Note 3.—It is often easier to clear an equation of fractions gradually. In 3, we began by bringing the two fractions having simple denominators to the same side, and then reducing them to a simple fraction, $\frac{5}{12}$; in 4, by reducing the fractions on the left-hand side to a single fraction, and then those on the right-hand side to another.

Note 4.—If the coefficients are decimals, express them as vulgar fractions, and proceed as before.

5. Solve

$$\cdot 75x - 1\cdot 5 = \cdot\dot{3}x + 3\cdot 5$$
$$\frac{75x}{100} - \frac{15}{10} = \frac{3x}{9} + \frac{35}{10}$$
$$\frac{3x}{4} - \frac{3}{2} = \frac{x}{3} + \frac{7}{2}$$
$$9x - 18 = 4x + 42$$
$$9x - 4x = 42 + 18$$
whence $x = 12$

6. Solve

$$1\cdot 5 + \frac{\cdot 75x - 2\cdot 5}{10} = \cdot 2x$$
$$\frac{15}{10} + \frac{75x - 250}{1000} = \frac{2x}{10}$$
$$\frac{3}{2} + \frac{3x - 10}{40} = \frac{x}{5}$$
$$60 + 3x - 10 = 8x$$
$$3x - 8x = -60 + 10$$
whence $x = 10$

7. Solve $\dfrac{x}{a-b} = \dfrac{2ab - 4b^2}{a^2 - b^2} + \dfrac{x}{a+b}$

Transposing, $\dfrac{x}{a-b} - \dfrac{x}{a+b} = \dfrac{2ab - 4b^2}{a^2 - b^2}$

Multiplying by L.C.D., $(a+b)x - (a-b)x = 2ab - 4b^2$

Collecting coefficients, $(a + b - a + b)x = 2b(a - 2b)$

$$2bx = 2b(a - 2b)$$

Dividing by $2b$, $\underline{x = a - 2b}$

Exercise XXXV. (A).

1. $\dfrac{x}{2} = 9$; $\dfrac{3x}{4} = 6$; $\dfrac{5x}{4} = 2\frac{1}{2}$; $\dfrac{3x-5}{10} = 4$; $3(x-5) = \dfrac{9}{4}$; $\dfrac{3x}{8} = \dfrac{3}{4}$.

2. $\dfrac{7x}{8} = \dfrac{216 - 3x}{12}$; $\dfrac{10x + 3}{7} = \dfrac{4x + 5}{3}$; $\dfrac{x}{3} - \dfrac{x}{4} = 2$; $\dfrac{x}{8} + 3 = \dfrac{x}{5}$.

3. $\dfrac{x}{6} + \dfrac{x}{8} - 7 = 0$; $2 + \dfrac{x}{14} = \dfrac{x}{6} - 2$; $\dfrac{x}{2} - \dfrac{x}{3} + \dfrac{x}{6} = 4$; $\dfrac{x}{6} - \dfrac{x}{4} = 2 - \dfrac{x}{9}$.

4. $\dfrac{2x}{3} - \dfrac{x + 4}{7} = \dfrac{8x - 3}{21}$; $x + \dfrac{x}{6} + \dfrac{x}{9} = 23$; $\dfrac{5x}{6} - \dfrac{4x}{9} = \dfrac{8x}{15} - \dfrac{13}{20}$.

5. $\dfrac{8x}{9} - \dfrac{7x}{12} - \dfrac{x}{6} = 3\frac{3}{4}$; $\dfrac{x}{2} + \dfrac{x}{5} = x - 6$; $\dfrac{3x}{4} + 5 = x + 2$.

6. $\dfrac{x - 3}{3} - 9 = \dfrac{x}{2} - x$; $\dfrac{x}{3} - \dfrac{x - 4}{8} = 11 - \dfrac{x}{12}$; $x + \dfrac{3x}{5} - \dfrac{4x}{8} + \dfrac{3x}{10} = 24$.

7. $\dfrac{6x}{7} - 9 = \dfrac{5x}{8} - 2\frac{1}{2}$; $\dfrac{9x}{10} - 3\frac{1}{8} = \dfrac{3x}{8} + 5\frac{4}{5}$; $\dfrac{x}{4} + \dfrac{5x}{6} - 3 = \dfrac{7x}{9} + \dfrac{2}{5}$.

8. $x - 9 - \dfrac{x + 9}{5} = \dfrac{x}{8}$; $3x - \dfrac{7x + 5}{6} = \dfrac{4x}{5} - 6$.

9. $\dfrac{2x}{3} - \dfrac{7x}{12} - 2\frac{1}{7} = \dfrac{5x}{7} - \dfrac{3x}{4}$; $\dfrac{4x}{9} - \dfrac{7x}{12} - \dfrac{19}{24} = 4 - \dfrac{5x}{6} + \dfrac{3x}{8}$.

10. $\frac{1}{2}(3x+5)+\frac{1}{3}(3+5x)=16$; $\frac{1}{3}(5x+8)-1\frac{1}{3}=\frac{1}{12}(7x+4)$.

11. $\frac{1}{3}(x+5)-\frac{1}{4}(x-2)=\frac{1}{5}(x+8)$; $\frac{4}{5}(2x+4)-\frac{1}{7}(5x+6)=x-2\frac{2}{3}$

12. $3 \cdot \frac{x-1}{2} - 2 \cdot \frac{2-x}{3} = 2x$; $\frac{x-2}{1\frac{1}{3}} - \frac{x+2\frac{1}{2}}{3} = \frac{x-4}{4}$.

13. $\frac{4}{3x-5}=\frac{8}{5x-3}$; $\frac{4x}{5x+9}=\frac{3x}{5x-7}$; $\frac{x-6}{x}=\frac{x-10}{x-6}$; $\frac{x+1}{x-1}=1+\frac{4}{x}$.

14. $\frac{6x}{3x+4}=\frac{2x+1}{x+3}$; $\frac{2x+5}{3x+4}=\frac{4x+5}{6x+1}$; $\frac{3}{x-1}-\frac{2}{x+1}=\frac{7}{x^2-1}$.

15. $\frac{x}{4}=\frac{x^2+2x}{4x+3}-\frac{1}{4}$; $\frac{2x}{x-3}-\frac{x-3}{x-6}=1$; $\frac{3}{x}+\frac{x(x-3)}{x-2}=x-1$.

16. $\frac{3}{2x-6}+\frac{1}{12}=\frac{x}{3(x-3)}$; $\frac{1}{2(x-3)}-\frac{1}{3(x-2)}=\frac{x-5}{(x-2)(x-3)}$.

17. $\frac{x}{x-2}+\frac{3x}{x+2}=4$; $x-\frac{x}{x-1}=\frac{x(x-1)}{x+1}$; $\frac{1}{x-1}+\frac{2}{x-5}=\frac{3}{x+7}$.

18. $\frac{3x+4}{3}-\frac{5x-4}{x+5}=\frac{4x-5}{4}$; $\frac{3x+5}{6}-\frac{3x+2}{2(x-1)}=\frac{4x-7}{8}$.

19. $\frac{x+6}{2x+8}+\frac{2-3x}{15}=\frac{3-2x}{10}$; $\frac{5x+4}{1-9x}+\frac{7x+1}{7}=\frac{5x-3}{5}$.

20. $\frac{3}{8}(3x+1\frac{1}{2})-\frac{3}{5}(x-2\frac{1}{2})+\frac{2}{3}(2x-3\frac{3}{4})=18\frac{1}{2}$.

21. $\frac{1}{2}\left[x-\frac{1}{2}\{x-\frac{1}{2}(x-1)\}\right]+3=\frac{1}{4}(x+7)+5$.

22. $\frac{1}{2}\left[x-\frac{1}{7}\{2x+\frac{1}{4}(x-1)\}\right]-\frac{1}{4}(x+5)=\frac{1}{4}(x-9)-3$.

23. $\frac{1}{x-3}-\frac{1}{x-5}=\frac{2}{x-6}-\frac{2}{x-7}$; $\frac{x}{x-1}-\frac{x-1}{x-2}=\frac{1}{x-4}-\frac{1}{x-5}$.

24. $\frac{x-7}{x-8}-\frac{x-6}{x-7}=\frac{x-4}{x-5}-\frac{x-3}{x-4}$; $\frac{x-1}{x-2}-\frac{x-3}{x-4}=\frac{x-2}{x-3}-\frac{x-4}{x-5}$.

25. $\cdot 75x+3=1\cdot 5x-3$; $\cdot 3x+2\cdot 5=1\cdot 25x-3$.

26. $3(x+5)-\cdot 6x=2\cdot 75x+11\cdot 5$; $3+\cdot 75x=2\cdot 375x-23$.

27. $2\cdot 95x-1\cdot 125=\cdot 6x+4\cdot 75$; $\cdot 15x+3\cdot 2=3\cdot 16x-11\cdot 85$.

28. $\frac{\cdot 1x-\cdot 5}{\cdot 5}-10=\frac{x+5}{2\cdot 5}-15$; $\frac{\cdot 5x+9\cdot 5}{2\cdot 5}=3+\cdot 25x$.

29. $\frac{\cdot 25x-2}{1\cdot 25}+\frac{\cdot 5+\cdot 125x}{\cdot 5}+\frac{x-1}{7}=7-\frac{2\cdot 3-\cdot 1x}{\cdot 5}$.

30. $\frac{1\cdot 9x+2\cdot 6}{2\cdot 25}-\frac{4\cdot 75+2\cdot 25x}{3\cdot 5}=5\cdot 75-\frac{11\cdot 5x}{2}$.

31. $\frac{x}{a} = b$; $\frac{3x}{4m} = 6n$; $\frac{3x}{a} = \frac{3b}{c}$; $5(x - m) = \frac{15n}{4}$.

32. $\frac{x}{a} - \frac{x}{b} = \frac{1}{ab}$; $m + \frac{mx}{n} = a + \frac{ax}{b}$; $\frac{x}{a} - 1 = \frac{x}{b} - \frac{x}{c}$.

33. $\frac{x}{m+n} + 2n = \frac{x}{m-n}$; $\frac{x-a}{x+b} - \frac{x-b}{x+a} = \frac{abx}{(x+a)(x+b)}$.

34. $\frac{2x-b}{2} = \frac{(x-a)^2}{x-b}$; $\frac{1}{x+2a} - \frac{1}{x+4a} = \frac{1}{x+8a} - \frac{1}{x+10a}$.

35. $\frac{2x}{a-b} - a = \frac{x}{a+b} + 3b$; $8\left(\frac{x}{c}+1\right) = 9\left(\frac{x}{c}-1\right)$.

36. $\frac{a}{bx} + \frac{b}{cx} + \frac{c}{ax} = p$; $ab + x(a+b) - \frac{a^2c}{b} = a(b+x)$.

37. $\frac{x-a}{a} + \frac{x-b}{b} = c - 2$; $\frac{ax-1}{ac} + \frac{bx-1}{ab} = \frac{1-cx}{bc}$.

38. $\frac{x-p}{q} + \frac{x-q}{r} + \frac{x-r}{p} = \frac{x-(p+q+r)}{pqr}$.

39. $\frac{x-a}{a} + \frac{x-b}{b} + \frac{x-c}{c} = \frac{a(b+c)x}{abc}$.

40. $2x(1+2c) + \frac{ac+c}{c} - \frac{a^2}{a+b} = a + b + \frac{ab+a+b}{a+b} + 2x$.

Problems resulting in Fractional Equations.

Examples, with Notes.

69. (b) 1. A gave one-half of his money to B, and one-third of it to C; B received £12, 10s. more than C. How much money had A?

Let x = amount of money A had,

then $\frac{x}{2}$ = " " " B received,

and $\frac{x}{3}$ = " " " C "

$\therefore \frac{x}{2} - \frac{x}{3} = 12\frac{1}{2}$, whence x = £75.

Note 1.—If a piece of work be done in x days, hours, &c., the part of it done in one day, hour, &c., is $\frac{1}{x}$. Or, generally,

$$\frac{\text{Work}}{\text{Time}} = \text{Rate}; \frac{\text{Work}}{\text{Rate}} = \text{Time}; \text{ Time} \times \text{Rate} = \text{Work}.$$

This applies to walking distances, filling cisterns, &c.

2. A can reap a field in 10 days, B in 15 days. How long will they take to reap it if they work together?

Let x = No. of days both take: then $\frac{1}{x}$ = part both do in 1 day;

but $\frac{1}{10}$ = part A does in 1 day; $\frac{1}{15}$ = part B does in 1 day;

$\therefore \frac{1}{10} + \frac{1}{15} = \frac{1}{x}$; $\therefore 15x + 10x = 150$; $\therefore x = 6$.

Note 2.—(1) The minute-hand of a clock travels 12 times as fast as the hour-hand; (2) When the hour strikes, the minute-hand is always, except at 12 o'clock, 5, 10, 15, &c. minutes before the hour-hand.

3. When are the hands of a watch together between 4 and 5?

Let x = No. of minutes travelled by short hand since 4 o'clock.
Then $12x$ = " " long hand " "
Also $x + 20$ = " " long hand " "
for it has travelled the short hand's distance and 20 min. more;
$\therefore 12x = x + 20$; $\therefore 11x = 20$; $\therefore x = 1\frac{9}{11}$, $x + 20 = 21\frac{9}{11}$;
that is, the hands are together at $21\frac{9}{11}$ minutes past 4.

4. When are the hands of a watch at right angles between 4 and 5? (That is, when is the long hand 15 minutes *behind* or 15 minutes *before* the short one between 4 and 5?)

Let x = No. of min. travelled by short hand since 4 o'clock.
Then $12x$ = " " long hand " "
Also $x + 20 - 15$ = " " long hand " if behind.
Or $x + 20 + 15$ = " " long hand " if before.
$\therefore 12x = x + 20 - 15$; $\therefore x = \frac{5}{11}$. $x + 20 - 15 = 5\frac{5}{11}$.
Or $12x = x + 20 + 15$; $\therefore x = 3\frac{2}{11}$, $x + 20 + 15 = 38\frac{2}{11}$;
that is, they are at right angles at $5\frac{5}{11}$, or $38\frac{2}{11}$ minutes past 4.

Note 3.—It is often better to make x stand, not for the unknown quantity, but for some other quantity from which it may be easily deduced.

5. Edinburgh is $20\frac{1}{4}$ miles from North Berwick; A sets out from Edinburgh, at the rate of 5 miles in 2 hours, at the same time as B sets out from North Berwick, at the rate of 21 miles in 8 hours. Where will they meet?

Let x = No. of hrs. A and B must travel before they meet.

A walks $\frac{5}{2}$ miles, and B $\frac{21}{8}$ miles an hour respectively.

$\therefore$ A and B approach by $(\frac{5}{2} + \frac{21}{8})$ miles in one hour;

$\therefore$ " " $(\frac{5}{2} + \frac{21}{8})x$ " x hours.

But A and B meet, or walk $20\frac{1}{2}$ miles, in x hours;

$$\therefore \frac{5x}{2} + \frac{21x}{8} = 20\tfrac{1}{2}, \text{ whence } x = 4 \text{ hours.}$$

In four hours, A walks $\frac{5}{2} \times 4 = 10$ miles;

$\therefore$ A and B meet 10 miles from Edinburgh.

Exercise XXXV. (B).

1. Find (1) the number whose half is 10 more than its seventh; (2) the number whose third and fourth parts added together make 21; (3) the number whose fifth, sixth, and tenth parts together taken from 28 leave no remainder.

2. Divide £38 between A and B, so that three-fourths of A's share shall equal five-sixths of B's; (2) £50 between A and B, so that two-thirds of A's share shall exceed three-fifths of B's by £8.

3. Find the number whose third, fourth, and sixth parts together exceed five-ninths of itself by 7.

4. A basket weighs 3 lbs. A fish in it weighs 7 lbs. and half its own weight. Find the weight of the whole.

5. A wins from B half as many marbles as he already has; from C, two-thirds of the number he wins from B, and 2 from D. He then has twice as many as at first. How many has he now?

6. How old was a lad who said: 'When I am 9 years older, I shall be half as old as my father, who is 3 times as old as I?'

7. Divide £1350 among three persons, so that the first may have two-thirds of the second's share, and the second three-fourths of the third's share.

8. Find two numbers whose sum is 35, and of which the greater divided by the less gives 4 as a quotient and 5 over.

9. Divide £100 between A and B, giving B the greater, so that half of A's share added to two-thirds of B's shall be 3 times as much as the difference between their shares.

10. One pipe can fill a cistern in 9 hours, another in 12, and a third in 18. In what time will all together fill it?

11. A pipe fills a bath in 45 minutes; another empties it in an hour. If both are open, in what time will it be filled?

12. A can build a wall in 10 days; B, in 15 days; A, B, and C together, in 4 days. In what time could C alone do it?

13. A works twice as fast as B, and B thrice as fast as C. Together they dig a well in 5 days. In what time could A do it?

14. A bought eggs at 5d. a dozen, and lost 1s. by selling equal quantities of them at 3 a penny and 4 a penny. How many had he?

15. When are the hands of a watch together (1) between 1 and 2; (2) between 3 and 4; (3) between 6 and 7; (4) between 8 and 9?

16. When are the hands of a watch (1) at right angles between 5 and 6; (2) opposite each other between 4 and 5; (3) 20 minutes apart between 7 and 8; (4) 5 minutes apart between 2 and 3?

17. A person saves one-fourth of his income. If his income were half as much more, and he spent £20 more, what he saved would then be half what he spent. Find his income at present.

18. A has 3s. for every 2s. B has; A gives B 14s., and now B has 5s. for every 4s. A has. How much has each now?

19. Half a gallon more than half the number of gallons in a cask are sold, and then half a gallon more than half the remainder; and then it is empty. How many gallons were in it?

20. Divide 36 into 4 parts, so that the first *plus* 2, the second *minus* 2, twice the third, and half the fourth may all be equal.

21. Find two consecutive even numbers, such that three-sevenths of the greater exceeds one-fourth of the less by 18.

22. A, B, and C have a certain sum between them. A and B together have $\frac{7}{12}$ of the whole; A and C, $\frac{2}{3}$; and B and C, £90. How much money has each?

23. An estate consists of three-fifths arable land, one-fifth pasture, one-tenth moorland, and 120 acres of woodland. Find the extent of the estate.

24. Divide £10, 15s. among 8 men, 10 women, and 12 boys, so that each woman may get three-fourths of a man's share, and each boy half a man's share.

25. If 20 oxen and 100 sheep cost £475, and each sheep cost

5s. less than one-fifth of the price of an ox, what is the price of an ox and of a sheep?

26. A number consists of two digits, the right digit being half the left. If 36 be deducted from the number, the digits are reversed. Find the number.

27. A number consists of two digits, the right digit being three times the left. If 54 be added to the number, the digits are reversed. Find the number.

28. The successful candidate at an election had a majority of 867 votes; but if 20 per cent. of his supporters had voted for his opponent, he would have been in a minority of 411. How many voted for each candidate?

29. The united population of two burghs is 19,052; $\frac{3}{4}$ of the population of one burgh equals $\frac{8}{7}$ of the population of the other. Find the population of each.

30. After paying away half of the money in his purse, then one-third of the remainder, and then three-fourths of what still remained, a man had 2s. 6d. left. How much money had he in his purse at first?

31. A and B together can do a piece of work in 6 days, B and C in $8\frac{4}{7}$ days, and A and C in $6\frac{2}{3}$ days. In how many days would each do it separately?

32. A person has one-fourth of his money invested at 3 per cent., one-half at $3\frac{1}{2}$ per cent., and the remainder at 4 per cent. His annual income from interest is £350. Find the amount of each investment.

33. A father is a years old, and is $2\frac{1}{2}$ times as old as his son; in how many years will he be twice as old?

34. A man walks 30 miles in 10 hours. For 3 hours he walks a miles an hour, and for 2 hours b miles an hour. Find his average rate per hour during the remainder of the journey.

Simultaneous Simple Equations.

70. Equations are called *Simultaneous* when each unknown quantity in one of them has the same value that it has in the others.

If an equation contain more than one unknown quantity, we must, in order to solve it, have it expressed in as many different forms as it has unknown quantities.

71. To Solve Equations containing Two Unknown Quantities.—There are several methods: (1.) *Elimination*, (2) *Substitution*, (3) *Comparison*; the object aimed at in each being to reduce the two equations to a single one containing only one unknown quantity. The first method is generally the simplest.

I. By *Elimination*.—Rule. (1) Multiply one or both equations, if necessary, by such numbers as will make the coefficients of one of the unknown quantities in each equal; (2) *add* the resulting equations if the equal coefficients have *different* signs, or *subtract* the one from the other if they have the *same* sign, and the result will be an equation containing only one unknown quantity; (3.) find the value of this quantity; (4.) put its value in place of itself in either of the original equations, and thus find the value of the other quantity.

Examples.

1. Find x and y when (1) $3x+7y=53$, and (2) $6x+5y=61$.

Multiplying (1) by 2, $\left.\begin{array}{r} 6x+14y=106 \\ 6x+\ 5y=\ 61 \end{array}\right\}$ $\therefore$ subtracting,
but (2), $9y=45$, $\therefore y=5$.

But $3x+7y=53$, $\therefore 3x+35=53$, $\therefore 3x=18$, $\therefore x=6$.

2. Find x and y when (1) $5x-4y=43$, and (2) $7y+35=3x$.

Mult. (1) by 3, $\left.\begin{array}{r} 15x-12y=\ \ 129 \\ -15x+35y=-175 \end{array}\right\}$ $\therefore$ adding,
" (2) by 5, $23y=-46$, $\therefore y=-2$.

But $5x-4y=43$, $\therefore 5x+8=43$, $\therefore 5x=35$, $\therefore x=7$.

II. By *Substitution*.—Find the value of x in terms of y from one of the equations, and substitute this value for x in the other.

Find x and y when (1) $x-3y=4$, (2) $3x-4y=22$.

From (1), $x=4+3y$; $\therefore$ from (2), $3(4+3y)-4y=22$;

that is, $12+9y-4y=22$, $\therefore y=2$; $\therefore$ from (1), $x=10$.

III. By *Comparison*.—Find the value of x in terms of y from each equation, and then equate these values of x.

Find x and y when (1) $2x-3y=2$, (2) $3x+4y=20$.

From (1), $x=\dfrac{2+3y}{2}$; from (2), $x=\dfrac{20-4y}{3}$;

$\therefore \dfrac{2+3y}{2}=\dfrac{20-4y}{3}$, $\therefore 6+9y=40-8y$; whence $y=2$, $x=4$.

Exercise XXXVI.

1. $x + y = 8$
   $x - y = 2.$

2. $x + y = 13$
   $y - x = 5.$

3. $5x + 8y = 71$
   $5x + 3y = 36.$

4. $3x + 5y = 49$
   $3x - 2y = 14.$

5. $7x + 8y = 97$
   $5y + 7x = 79.$

6. $6x + 5y = 42$
   $3x + 2y = 18.$

7. $8x + 4y = 72$
   $3x + 2y = 31.$

8. $2x + 7y = 46$
   $3x + 5y = 47.$

9. $4x + 7y = 40$
   $5x + 6y = 39.$

10. $6x + 11y = 21$
    $9x + 7y = 3.$

11. $16x + 17y = 42$
    $24x + 19y = 50.$

12. $4x + 5y = 56$
    $6x - 7y = 26.$

13. $13x + 6y = 67$
    $17x - 9y = 82.$

14. $7x - y = 20$
    $18x + 19y = 73.$

15. $25x - 16y = 97$
    $15x + 8y = 199.$

16. $24x - 17y = 60$
    $36x - 31y = 24.$

17. $27x - 3y = 24$
    $25x + 4y = 90.$

18. $8x - 7y = -3$
    $6y - 5x = 10.$

19. $8x + 5y = 13$
    $9y - 4x = 97.$

20. $4x - 5y = 22$
    $7y - 3x = -23.$

21. $5y - 7x = 16$
    $3x - 7y = -2.$

22. $7y - 18x = 40$
    $36x + 9y = 12.$

23. $3y + 12x = 0$
    $3 - 8x = y.$

24. $\frac{x}{3} + \frac{y}{4} = 6$
    $\frac{x}{3} + \frac{y}{6} = 5.$

25. $\frac{x}{2} + y = 10$
    $x + \frac{y}{3} = 10.$

26. $3x + \frac{y}{5} = 39$
    $2y - \frac{x}{4} = 27.$

27. $\frac{x + y}{4} + 2 = 6$
    $\frac{x - y}{3} + 3 = 5\frac{2}{3}.$

28. $\frac{3x - 4}{5} = 3y - 2$
    $\frac{2x + 3y}{2} = 10 + \frac{y}{2}.$

29. $2x + \frac{y - 4}{3} = 7$
    $3y - \frac{7 - x}{5} = 38.$

30. $\frac{2x}{6} + y = 19$
    $7x - \frac{6y - 4}{5} = 89.$

31. $\frac{x}{5} + 6 = \frac{y}{3} + 4$
    $\frac{y - x}{2} + 2x = 21.$

32. $\frac{x + y}{5} + \frac{x - y}{4} = 3$
    $13y - 3x = 18.$

33. $\frac{2}{x} + \frac{3}{y} = 4$
    $\frac{4}{x} + \frac{2}{y} = 4.$

34. $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}, \quad \frac{2}{x} + \frac{1}{2y} = \frac{1}{15}.$

72. To Solve Equations containing Three Unknown Quantities.—(1) From any two of the equations find one containing only two unknown quantities, as in sect. 71; (2) from the third and either of the former two find another containing the same two unknown quantities; (3) from the two equations thus found, find one containing only one unknown quantity.

Example.

Find the values of x, y, and z, when

(1) $3x-2y+4z=23$; (2) $2x+y-5z=4$; (3) $5x-3y+2z=26$.

Multiplying

$$\left.\begin{array}{lr} (1) \text{ by } 2, & 6x-4y+8z=46 \\ (2) \text{ by } 3, & 6x+3y-15z=12 \end{array}\right\} \begin{array}{l} \therefore \text{subtracting,} \\ -7y+23z=34 \quad \ldots (4) \end{array}$$

$$\left.\begin{array}{lr} (2) \text{ by } 5, & 10x+5y-25z=20 \\ (3) \text{ by } 2, & 10x-6y+4z=52 \end{array}\right\} \begin{array}{l} \therefore \text{subtracting,} \\ 11y-29z=-32 \quad \ldots (5) \end{array}$$

$$\left.\begin{array}{lr} (4) \text{ by } 11, & -77y+253z=374 \\ (5) \text{ by } 7, & 77y-203z=-224 \end{array}\right\} \begin{array}{l} \therefore \text{adding,} \\ 50z=150;\ \therefore z=3. \end{array}$$

From (5), $11y-87=-32$, $\therefore y=5$;

from (1), $3x-10+12=23$, $\therefore x=7$.

Exercise XXXVII.

1. $2x+5y+8z=66$, $x+2y+3z=26$, $3x+7y+9z=82$.
2. $3x+4y+2z=17$, $2x+y-z=5$, $4x-3y+2z=13$.
3. $4x-3y+2z=10$, $5x-4y+z=10$, $2x+y-3z=11$.
4. $3x+2y-5z=1$, $5x+3y-6z=16$, $4x-y-z=19$.
5. $2x-4y-z=0$, $3x-7y+4z=8$, $5x-3y+2z=30$.
6. $3x+4y+5z=61$, $7x-9y+4z=-6$, $x+y+z=15$.
7. $3x+4y=26$, $2y-z=0$, $x-3y+2z=8$.
8. $\frac{x}{2}+\frac{y}{4}+\frac{z}{3}=11$, $\frac{x}{5}+y-\frac{z}{4}=7$, $x+\frac{y}{2}+z=26$.

Problems resulting in Simultaneous Equations.

Examples, with Notes.

Note 1.—A number of two figures may be represented thus, $10x+y$; if the figures be reversed, thus, $10y+x$; one of three

figures, thus, $100x + 10y + z$. For a figure in the first place means so many units; one in the second place, so many tens; and one in the third place, so many hundreds.

1. The sum of the two figures of a number is 7; if 9 be added to the number, the figures are reversed. Find the number.

Let x = figure in tens' place, y = figure in units' place.

Then $10x + y$ = the number, $10y + x$ = the number with figures reversed;

$$\left.\begin{array}{l} \therefore 10x + y + 9 = 10y + x; \ \therefore 9x - 9y = -9 \\ \text{and } x + y = 7 \quad ; \ \therefore 9x + 9y = 63 \end{array}\right\} \therefore 18x = 54.$$

Whence $x = 3$, $y = 4$, and the number is 34.

Or, let x = figure in tens' place, $7 - x$ = " units' " — as if it were an equation containing one unknown quantity.

Then $10x + (7-x) + 9 = 10(7-x) + x$; whence $x = 3$, $7-x = 4$.

Note 2.—A fraction may be represented thus, $\frac{x}{y}$.

2. What fraction becomes 1, if 1 be added to its numerator, and $\frac{1}{2}$ if 6 be added to its denominator?

Let $\frac{x}{y}$ = the fraction; then $\frac{x+1}{y} = 1$, and $\frac{x}{y+6} = \frac{1}{2}$;

$\therefore x - y = -1$, $2x - y = 6$; hence $x = 7$, $y = 8$, and the fraction is $\frac{7}{8}$.

Exercise XXXVIII.

1. Find (1) two numbers whose sum is 80, and whose difference is 6; (2) two numbers whose sum is 40, and of which twice the greater equals thrice the less; (3) two numbers whose sum is 54, and of which the one has to the other the ratio of 4 to 5.

2. Three cows and 5 horses cost £166; four cows and three horses cost £126. Find the price of a cow and of a horse.

3. Six men and 9 boys earn £5, 8s. in 4 days; and 4 men and 10 boys earn £3, 10s. in 3 days. Find the daily wage of each.

4. Six years ago A was 3 times as old as B, and 6 years hence A will be twice as old as B. Find their ages.

5. The ages of two persons are in the ratio of 4 to 5, but 8 years ago they were in the ratio of 3 to 4. Find their ages.

6. Find two numbers whose sum is 20, and of which twice the less is to thrice the greater as the square of the less to the square of the greater.

7. If A give B £5, B will have thrice as much as A. If B give A £5, A will have twice as much as B. How much has each?

8. How much tea at 3s. 4d. and 3s. 9d. per lb. respectively must I mix together, so as to have 30 lbs. worth 3s. 6d. each?

9. £1200 is invested partly at 5 per cent. and partly at 4 per cent.; and the whole interest received for it is £53. How much is invested each way?

$$\text{Interest} = \frac{\text{Principal} \times \text{Rate per cent.} \times \text{Time}}{100}.$$

10. A sum of money put out at simple interest, amounts to £3000 in 4 years, and to £3250 in 6 years. Find the sum and the rate per cent.

11. The sum of the two figures of a number is 8; if 18 be taken from the number, the figures are reversed. Find the number.

12. A number has two figures, the greater in the units' place. The sum of the figures is 6; and if the number be divided by their difference, the result is 12. Find the number.

13. Find a fraction which becomes $\frac{2}{3}$ if 1 be added to its numerator, and $\frac{1}{2}$ if 1 be added to its denominator.

14. What fraction becomes $\frac{4}{5}$ when 4 is added to both its numerator and denominator, and $\frac{2}{3}$ when 2 is taken from both?

15. A can buy a horse worth £33, if B give him $\frac{1}{3}$ of his money; B can buy it if A give him $\frac{1}{4}$ of his. How much has each?

16. A person spends 4s. altogether on apples at 4 a penny, and pears at 3 a penny. He sells half the apples and a third of the pears at the same rate for 19d. How many of each did he buy?

17. Twelve horses and 5 cows cost a certain sum; 3 horses and 10 cows cost half of it. How many horses alone, or cows alone, could be bought for it?

18. A train runs from A to B in a certain time; if it ran 4 miles an hour faster, it would take an hour less; if as much slower, it would take 84 min. more. How far is it from A to B?

19. A and B together reap a field in 2 days, and A works twice as fast as B. In what time could either do it alone?

20. A and B have together to pay a debt of £30. A could pay it if B would give him $\frac{3}{8}$ of his money, and B could if A would give him $\frac{4}{9}$ of his. How much has each?

21. When apples are selling at 3 for 2d., and oranges at 2 for 3d., a woman buys a certain number of each for 5s. 10d. Had she bought twice as many apples and half as many oranges, she would have paid 11d. less. How many of each did she buy?

TEST EXERCISES.

Note.—Exercises XI.-XX. contain questions for revisal of Parts I. and II., and will supply useful drill for pupils preparing for the junior certificate in mathematics at the University Local Examinations.

I.

1. Find the G.C.M. and the L.C.M. of
$$x^4 - 1 \text{ and } x^3 - 2x^2 - x + 2.$$

2. Simplify $\dfrac{x^3 - x^2y}{y^3 - x^2y} + \dfrac{x(x+y)}{(x-y)y} + \dfrac{2xy}{x^2 - y^2}$.

3. Reduce to its lowest terms $\dfrac{2x^2 - 5x + 2}{x^3 + 4x^2 - 4x - 16}$.

4. Solve (1) $\frac{1}{6}(2x - 1) + \frac{1}{12}(3x + 3) = \frac{1}{12}(7x - 17)$.
   (2) $\cdot 25x + 3\cdot 5 = \cdot 625x - 1$.

5. A purse contains sovereigns and half-crowns to the value of £12, 10s. The half-crowns number twice as many as the sovereigns. How many are there of each?

II.

1. Find the G.C.M. of
$$x^3 - 2x^2 + 3x - 6 \text{ and } x^4 - x^3 - x^2 - 2x.$$

2. Simplify (1) $\dfrac{a}{a - x} + \dfrac{3a}{a + x} - \dfrac{2ax}{a^2 - x^2}$; (2) $\dfrac{x^3 + a^3}{x^4 + a^2x^2 + a^4}$.

3. Simplify $\left(1 + \dfrac{x}{y}\right)\left(1 - \dfrac{x}{y}\right) \div \dfrac{y}{x^2 + y^2}$.

4. Solve (1) $\frac{1}{3}(x - 4) + (x - 1)(x - 2) = x^2 - 2x - 4$.
   (2) $5x + 4y = 34 = 6x - 2y$.

5. The difference of the squares of two consecutive even numbers is 60. Find the numbers.

III.

1. Reduce to its lowest terms $\dfrac{a^3 + 6a^2 + 11a + 6}{a^3 + 5a^2 + 6a}$.

2. Simplify $\dfrac{x^2 - x - 6}{x^2 - x - 20} \times \dfrac{x^2 - 12x + 35}{x^2 + 10x + 16} \times \dfrac{x^2 + 12x + 32}{x^2 + 2x - 15}$.

3. Simplify $\frac{x^2+3x+2}{x^2+2x+1}+\frac{x^2-1}{x^4-2x^2+1}-\frac{x^3+x}{x^4-1}$.

4. Solve the equation,

$$(1)\ \frac{x-1}{2}+\frac{x-5}{4}+\frac{x-13}{8}+\frac{x-29}{16}=x-8.$$

$$(2)\ (x+5)^2-32=(9x+1)\left(\frac{x}{9}+1\right).$$

5. Find a number such that its half increased by 9 is seven times its eighth part.

IV.

1. Find the G.C.M. and the L.C.M. of $x^4+2x^3-7x^2-8x+12$ and $x^3-4x^2-7x+10$.

2. Simplify $\frac{a}{a-x}-\frac{x}{a+x}+\frac{2x^2}{x^2-a^2}$.

3. Divide $\frac{a^4-x^4}{a^2+2ax+x^2}$ by $\frac{a^2-2ax+x^2}{a^2-x^2}$.

4. Solve (1) $\frac{2}{2x-3}+\frac{1}{x-2}=\frac{6}{3x+2}$.

   (2) $\frac{1}{2}(x+a)+\frac{1}{3}(x+2a)+\frac{1}{4}(x+3a)=16$.

5. A father's age exceeds his son's by 31 years, and is as much below 60 as his son's is above 19. Find the age of each.

V.

1. Find (1) the G.C.M. of x^2-3x+2 and x^2-6x+5.

   (2) the L.C.M. of $x+1$, x^3+1, x^4-1, and x^6-1.

2. Reduce to its lowest terms $\frac{m^3+3m^2-20}{m^4-m^2-12}$.

3. Simplify $\frac{x+1}{x^3+1}-\frac{x-1}{x^3-1}-\frac{2x}{x^4+x^2+1}$.

4. Solve (1) $\cdot 6x-\frac{\cdot 09x-\cdot 05}{\cdot 5}=\cdot 2x+8\cdot 9$.

   (2) $2x+7y=13$; $6y-5x=38$.

5. The length of a floor exceeds the breadth by 4 feet; if each were increased by a foot, the area would be increased by 27 square feet. Find its dimensions.

VI.

1. Simplify $\dfrac{(a^3+b^3)(a^2-b^2)}{(a^2-ab+b^2)(a-b)}$.
2. Find the G.C.M. and the L.C.M. of x^6+y^6 and $x^4+x^2y^2+y^4$.
3. Find the sum of $\dfrac{1}{2a+b}$, $\dfrac{1}{2a-b}$, and $\dfrac{3a}{b^2-4a^2}$.
4. Solve (1) $\dfrac{x}{x-1}+\dfrac{2(x+2)}{x^2-1}=\dfrac{x+1}{x-1}$.

   (2) $(x-m)^2-(x-n)^2=mn(m-n)$.
5. When A married he was one-third older than his wife, but in eight years he was only one-fourth older. What were their ages when they were married?

VII.

1. Find the G.C.M. of $a^3-23a+10$ and $5a^3-23a^2+4$.
2. Simplify $\dfrac{x}{x+1}-\dfrac{x+1}{x-1}+\dfrac{2(x+2)}{x^2-1}$.
3. Divide $\dfrac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}$ by $\dfrac{x^4-y^4}{2xy(x-y)}$.
4. Solve (1) $(x-1)(x+2)+(x-3)(x+4)=2(x-1)(x+1)$.

   (2) $\dfrac{a}{x}+\dfrac{b}{y}=1=\dfrac{b}{x}-\dfrac{a}{y}$.
5. A has £15 more than B, and £25 more than C, and between them they have £125. How much has each?

VIII.

1. Reduce to its lowest terms $\dfrac{x^3+2x^2-8x-16}{x^3+3x^2-8x-24}$.
2. Simplify $\dfrac{2x-5}{4x^2-1}+\dfrac{5}{2x-1}-\dfrac{3}{x}$.
3. Find the product of

   $\dfrac{a^2-b^2}{a^2-8ab+7b^2}$, $\dfrac{a^2-49b^2}{a^3+b^3}$, and $\dfrac{a^2+2ab+b^2}{a^2+8ab+7b^2}$.
4. Solve (1) $\dfrac{x-2}{2}+\dfrac{x-1}{3}-2=\dfrac{7}{6}-\dfrac{2x}{3}$.

   (2) $\dfrac{(x+3)(x+1)(x-1)}{(x+4)(x+2)(x-3)}=1$.

5. If 8 lbs. of tea and 20 lbs. of sugar together cost £1, 9s., and a lb. of tea costs as much as 12 lbs. of sugar, find the price of a lb. of each.

IX.

1. Find the G.C.M. of
$a^2 + 2ab + b^2 - c^2$ and $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

2. Simplify $\dfrac{1}{a+2b} + \dfrac{1}{b-a} - \dfrac{3a}{(a+2b)^2}$.

3. Divide $\dfrac{a^2}{b^3} + \dfrac{1}{a}$ by $\dfrac{a}{b^2} - \dfrac{1}{b} + \dfrac{1}{a}$.

4. Solve (1) $\dfrac{6x+7}{15} - \dfrac{2x-2}{7x-6} = \dfrac{2x+1}{5}$.

(2) $3x + 5 : 9x + 2 :: 2 : 5$.

5. Divide 96 oranges between A, B, and C, so that B may have twice as many as C, and A five times as many.

X.

1. Find the G.C.M. and the L.C.M. of
$6a^3 + a^2b - 11ab^2 - 6b^3$ and $6a^3 + 11a^2b - ab^2 - 6b^3$.

2. Simplify $\dfrac{1}{2x(x+y)} + \dfrac{1}{2x(x-y)} + \dfrac{2}{y^2-x^2}$.

3. Simplify $\left(\dfrac{a+1}{a-1} - \dfrac{a-1}{a+1}\right) \div \left(\dfrac{a+1}{a-1} + \dfrac{a-1}{a+1}\right)$.

4. Solve (1) $\dfrac{1}{x-1} - \dfrac{1}{x-3} = \dfrac{1}{x-2} - \dfrac{1}{x-4}$.

(2) $\dfrac{x^2-a^2}{x+a} + \dfrac{x^2-b^2}{x+b} + \dfrac{x^2-c^2}{x+c} = 2(a+b+c)$.

5. There is a number such that the square of the next higher number exceeds the square of the next lower number by 120. Find the number.

XI.

1. When $a = 4$, $b = 3$, $c = 2$, find the value of
$$\frac{a-b-c}{(a-b)(a-c)} + \frac{b-c-a}{(b-c)(b-a)} + \frac{c-a-b}{(c-a)(c-b)}.$$

2. Simplify $\dfrac{(x^3 + 6x^2 + 11x + 6)(x^2 - 4x + 3)}{(x^2 + 3x + 2)(x^2 - 9)}$.

3. Find the G.C.M. of
$6n^3 + 16n^2 - 12n + 2$ and $15n^3 - 5n^2 + 15n - 5$.
4. Find the sum of $\dfrac{1}{(x+y)^2} + \dfrac{1}{(x-y)^2} + \dfrac{4}{x^2-y^2}$.
5. Solve (1) $(x+4)(x+2)+(x+6)(x+3)+(x+2)(x+1) = 244 + 3x^2$.
(2) $\frac{1}{2}x + \frac{2}{3}y = 7$; $\frac{1}{10}x + \frac{1}{12}y = 1$.
6. A father is 48 years of age, and his son 22 years. In how many years will the son's age be half that of the father?

XII.

1. Multiply out $(x - \overline{b+c})(x - \overline{c+a})(x - \overline{a+b})$.
2. Show that $\dfrac{a^2+bc}{(a-b)(a-c)} + \dfrac{b^2+ca}{(b-c)(b-a)} + \dfrac{c^2+ab}{(c-a)(c-b)} = 2$.
3. Reduce to its lowest terms $\dfrac{a^2-b^2-c^2+2bc}{a^2+2ab+b^2-c^2}$.
4. Divide $\dfrac{1}{x^2} - \dfrac{1}{y^2} - \dfrac{1}{z^2} + \dfrac{2}{yz}$ by $\dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z}$.
5. Solve (1) $\frac{1}{3}(x + 1\frac{1}{2}) - 2 = \frac{1}{6}(2x - 3\frac{1}{3}) - 1\frac{1}{3}$.
(2) $x(x-a) + x(x-b) = 2(x-a)(x-b)$.
6. A number consists of two digits, the second being $3\frac{1}{2}$ times the first, and if 45 be added to the number its digits are reversed; find the number.

XIII.

1. Divide $(a+b)^2 - (c+d)^2 + (a+c)^2 - (b+d)^2$ by $a - d$.
2. Resolve into factors:
$a^2b^2 + 4ab - 12$; $x^2 - 2x - 3$; $8y^3 + x^3$; $a^2 + 2ab + b^2 - 9$.
3. Add together $\dfrac{1}{m+n} + \dfrac{1}{m-n} + \dfrac{4m-6n}{m^2-n^2}$.
4. Simplify $\left(\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r}\right)(p+q+r)pqr - (p+q)(q+r)(r+p)$.
5. Solve (1) $\dfrac{x^2-1}{x-1} + \dfrac{x^2-4}{x+2} + \dfrac{x^2-9}{x-3} = 12$.
(2) $\frac{1}{2}(x+1) + \frac{1}{3}(y+2) = 1 = \frac{1}{3}(x+1) + \frac{1}{2}(y+2)$.
6. A woman bought apples at 4 for 3d. She sold half of them at 3 for 2d. and half at a penny each, thereby gaining 6d. in all. How many did she buy?

XIV.

1. Divide $x^3 + y^3 + 8 - 6xy$ by $x + y + 2$, and verify the results when $x = 3$, $y = -1$.
2. Resolve into factors:
   $a^2x^2 - 11a^2x + 30a^2$; $m^6 - 64$; $9a^2 - (2b + c)^2$.
3. Find the G.C.M. of $x^4 + x^2 + 1$ and $x^4 - 2x^3 - x + 2$.
4. Simplify $\dfrac{\frac{a}{2} - \frac{a}{3}}{\frac{a}{2} + \frac{a}{3}} - \dfrac{\frac{b}{3} - \frac{b}{4}}{\frac{b}{3} + \frac{b}{4}} + \dfrac{\frac{c}{4} - \frac{c}{5}}{\frac{c}{4} + \frac{c}{5}}$.
5. Solve (1) $\dfrac{x-1}{3} - \dfrac{2x+1}{7} = \dfrac{3x-2}{21} - \dfrac{4}{3}$.
   (2) $(a + b)^2x + a^3 + b^3 = (a^2 - b^2)x + (a + b)^3$.
6. Edinburgh and Glasgow are 42 miles apart. A walks from Edinburgh, at the rate of 10 miles in 3 hours, and B from Glasgow, at the rate of 7 miles in 2 hours. When and where will they meet?

XV.

1. Simplify
   $(a^3 + 3a^2b + 3ab^2 + b^3)(a^2 + 2ab + b^2) - (a - b)^5 (a - b)^2$.
2. Divide $\frac{1}{4}m^3 + \frac{1}{72}mn^2 + \frac{1}{12}n^3$ by $\frac{1}{2}m^2 - \frac{1}{3}mn + \frac{1}{4}n^2$.
3. Reduce to their lowest terms:
   (1) $\dfrac{a^3b - b^4}{a^2b + ab^2 + b^3}$. (2) $\dfrac{x^4 - 1}{x^6 - 1}$.
4. Simplify $\dfrac{1 + \dfrac{2xy}{(x-y)^2}}{1 - \dfrac{2xy}{(x+y)^2}} \div \dfrac{\dfrac{x+y}{x}}{\dfrac{y-x}{y}}$.
5. Solve (1) $(3 - 4x)^2 + (4 - 4x)^2 = 2(5 + 4x)^2$.
   (2) $12x + 20y = 44$; $8x + 18y = 29$.
6. At what time between 3 and 4 o'clock is the minute-hand of a clock 20 minutes in advance of the hour-hand?

XVI.

1. If $a = 3$, $b = 2$, $c = 1$, find the value of
   $$\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ca}.$$
2. Divide $a^5 - a^4b + 2a^3b^2 - 2a^2b^3 + ab^4 - b^5$ by $a^2 + b^2$.

3. Simplify $\frac{2a}{(x-2a)^2} - \frac{x-a}{x^2-5ax+6a^2} + \frac{2}{x-3a}$.

4. Multiply $\frac{3x(x^2-y^2)}{5y(x^3-y^3)}$ by $\frac{10x(x-y)^3}{9y(x^2-2xy+y^2)}$.

5. Solve (1) $\frac{x}{4} - \frac{x+10}{5} + 4\frac{3}{4} = x - 1 - \frac{x-2}{3}$.

(2) $ax + 2b^2 = 2a^2 + bx$.

6. The sum of £5 is paid with half-sovereigns and half-crowns, and 19 coins are used; how many of each coin?

XVII.

1. Find the value of $(x-y-z)^2 + (y-x-z)^3 + (z-x-y)^2$ when $x = 2$, $y = 3$, $z = 5$.

2. Multiply $7x^2 - 3x^3 + x^4 - 6x + 4$ by $x^2 + 3x + 2$, and divide the product by $x^3 + 2x - x^2 + 4$.

3. Resolve $x^3 - 64y^3$; $2p^2 - 6pq + 4q^2$; $m^2 - n^2 - 1 + 2n$.

4. Simplify $\left(\frac{x^3}{a^3} - \frac{x}{a} + \frac{a}{x} - \frac{a^3}{x^3}\right) \div \left(\frac{x}{a} - \frac{a}{x}\right)$.

5. Solve (1) $\frac{4}{x-8} + \frac{3}{2x-16} - 1\frac{5}{24} = \frac{2}{3x-24}$.

(2) $\frac{1}{x} + \frac{2}{y} = a$; $\frac{3}{x} + \frac{4}{y} = b$.

6. Divide the number 57 into two such parts that the sum of the quotients obtained by dividing one part by 3 and the other by 6 may be equal to 16.

XVIII.

1. Divide $x^4 + (2b^2 - a^2)x^2 + b^4$ by $x^2 + ax + b^2$, and verify the result when $x = -1$, $a = 1$, $b = 2$.

2. Find the G.C.M. of $x^3 + 2x^2 - 3x - 10$ and $x^3 + 4x^2 - 5x - 14$.

3. Simplify $\frac{2}{x^2-4x+3} - \frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2}$.

4. Simplify $\frac{a-x}{a+x} \cdot \frac{a^2-x^2}{a^2+x^2} \cdot \frac{a^3+x^3}{a^3-x^3} \cdot \frac{a^4-x^4}{a^3x^3+x^6} \cdot \frac{a+x}{(a-x)^2}$.

5. Solve (1) $(6x + 14)(12x + 9) = (9x + 15)(8x + 10)$.

(2) $\frac{4x + 3{\cdot}5 + 2{\cdot}5y}{x+y+1} = 3$; $\frac{3{\cdot}5 + 2{\cdot}5y}{y+1} = 4$.

6. A regiment arranged before battle wanted 11 men to form a complete square; after battle, in which 100 men were lost, it formed a complete square 2 less in the side, with 9 men over. Find the original strength of the regiment.

XIX.

1. If $x = 5$, $y = -6$, $z = -4$, find the value of

   (1) $4x^2 + y^2 + z^2 + 4xy + 4xz + 2yz$.

   (2) $23x - 7[x - 6\{y + 5(3z + 12) + 7\} + 4]$.

2. Divide $x^8 + x^6 + x^4 + x^2 + 1$ by $x^4 - x^3 + x^2 - x + 1$.

3. Find the G.C.M. of $x^4 + x^2 + 1$ and $x^4 + 2x^3 + 3x^2 + 2x + 1$, and the L.C.M. of $(1 - 4x^2)^2$, $(1 + 2x)^3$, $(1 - 2x)^3$.

4. Divide $\left(1 + \dfrac{a^2 + 4b^2 - 9c^2}{4ab}\right)$ by $\left(\dfrac{a + 2b}{3c} - 1\right)$.

5. Solve (1) $21x - 7[x - 6\{12x + 7(3x - 12) - 9\} + 12] = 105$.

   (2) $\dfrac{x}{p} + \dfrac{x}{q} + \dfrac{x}{r} = p(q + r) + q(r + p) + r(p + q)$.

6. The wages of 12 men and 5 boys amount to £1, 17s. 6d. The wages of 12 boys exceed the wages of 7 men by 6d. Find the wages of a man and a boy.

XX.

1. Prove that

   $(b + c)^2 - a^2 + (c + a)^2 - b^2 + (a + b)^2 - c^2 = (a + b + c)^2$.

2. Find the product of $a^2 - b^2 + c^2 - d^2$ and $a^2 + b^2 - c^2 - d^2$; and divide $4y^4 - 9y^2 + 6y - 1$ by $2y^2 + 3y - 1$.

3. Resolve into factors:

   $5x^2 - 40x - 100$; $x^5 - xy^4$; $x^2 - y^2 + 2yz - z^2$.

4. Simplify (1) $\dfrac{n^2(n + 1)}{2} - \dfrac{(n - 1)n(n + 1)}{3}$.

   (2) $\dfrac{x^4 - y^4}{x^2 + 2xy + y^2} \div \dfrac{x^2 - 2xy + y^2}{x^2 - y^2}$.

5. Solve (1) $\dfrac{7 - x}{6 - x} - \dfrac{5 - x}{4 - x} = \dfrac{3 - x}{2 - x} + \dfrac{1 - x}{x}$.

   (2) $(x + a)^2 + (x + b)^2 + (x + c)^2 = 3x^2 + 2(ab + ac + bc)$.

6. The number of scholars absent from a school on a particular day is 8 per cent. of the number present, and is less than the number present by 874. Find the number of scholars on the roll.

STANDARD ALGEBRA—PART III.

INVOLUTION.*

73. By *Involution*, we mean *finding the powers of quantities.* Revise sect. 8, with special attention to its last sentence.

Where there is no bracket, an index refers only to the figure or letter to which it is attached.

Therefore, to indicate a power of a negative quantity, of a simple quantity with more than one factor, or of a compound quantity, these quantities must be bracketed.

Thus, $-a^2 = a\,.-a$, but $(-a)^2 = -a\,.-a$; $-2x^2 = -2\,.\,x^2$, but $(-2x)^2 = -2x\,.-2x$; $3ax^3 = 3a\,.\,x^3$, but $(3ax)^3 = 3ax\,.\,3ax\,.\,3ax$; $a - b^2 = a - b\,.\,b$, but $(a-b)^2 = (a-b)(a-b)$.

74. *Positive* quantities have *all* their powers *positive; negative* quantities have their *even* powers *positive*, their *odd* powers *negative*. Thus,

$$(-a)^2 = a^2,\ (-a)^4 = a^4;\ (-a)^3 = -a^3,\ (-a)^5 = -a^5.$$

The *even* powers of a quantity will therefore be the same, whether it be positive or negative. Thus,

$$(+a)^2 \text{ or } (-a)^2 = a^2;\ (2x)^4 \text{ or } (-2x)^4 = 16x^4;$$
$$(a-b)^2 \text{ or } (-a+b)^2 = a^2 - 2ab + b^2;\ (x-1)^4 = (1-x)^4.$$

75. To raise a power of a quantity to a higher power, *multiply the indices together*, and prefix the proper sign. Thus,

$$(a^2)^2 = a^2\,.\,a^2 = a^4 = a^{2\times 2};\ (a^4)^3 = a^4\,.\,a^4\,.\,a^4 = a^{12} = a^{4\times 3}$$
$$\{(-a)^2\}^2 = a^4;\ (-ab^2)^3 = -a^3b^6;\ (-2x^3)^6 = 64x^{18}.$$

But, to multiply one power by another, *add their indices*, sect. 29 (3). Thus, $a^2 \times a^3 = a^5$, but $(a^2)^3 = a^6$.

* See section 33 (2, and (4).

By attention to this the work of involution may often be shortened. Thus, we may find the 5th power of a quantity by multiplying its 3d by its 2d, for $a^5 = a^3 \times a^2$; its 7th, by multiplying its 4th by its 3d, for $a^7 = a^4 \times a^3$; its 4th, 6th, 8th, &c., by squaring its 2d, 3d, 4th, &c., for $a^4 = (a^2)^2$, $a^6 = (a^3)^2$, $a^8 = (a^4)^2$, &c.

Exercise XXXIX.

1. $(-x)^3$, $(-x)^4$, $(-x)^5$, $(2a)^3$, $(3a)^4$, $(-4a)^2$, $(-3b)^3$.

2. Add y^3 to $(-y)^3$, $(-a^5)$ to a^5, x^2 to $(-x)^2$, $(-z)^4$ to z^4, $2a^4$ to $(2a)^4$, $-3b^3$ to $(-3b)^3$, $1+4b^2$ to $(1+4b)^2$.

3. $(x^2)^2$, $(x^5)^2$, $(x^2)^3$, $(x^3)^3$, $(x^4)^3$, $(-x^3)^2$, $(-x^5)^4$, $(-x^3)^5$.

4. $(a^3 . a^2) + (a^3)^2$, $(a^5 . a^3) + (a^5)^3$, $(-a^3 . a^4) + \{(-a)^3\}^4$.

5. $(x+y)^2$, $(x-y)^2$, $(x+4)^2$, $(ax+3y)^2$, $(4x+5y)^2$, $(3x-2y)^2$.

6. $(a+x)^3$, $(m-n)^3$, $(x-3)^3$, $(1-3x)^3$, $(a^2+xy)^3$, $(a^2-ax)^3$.

7. $(x+z)^4$, $(x-3)^4$, $(2x-3y)^4$, $(a+y)^5$, $(1-x)^5$, $(2x+1)^5$.

8. $(m+n)^6$, $(x-1)^6$, $(1+x)^6$, $(3x+y)^6$, $(x-2y)^6$.

9. $(a+2b-c)^2$, $(a-2b+c)^2$, $(x^2+x-1)^3$, $(1-x-x^2)^4$.

76. The square of any quantity is made up of the squares of its terms, and twice the product of each term into each of the terms that follow it. Thus,

$$(a+b)^2 = a^2 + b^2 + 2ab;\quad (a-b)^2 = a^2 + b^2 - 2ab;$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc;$$
$$(a-b+c-d)^2 = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.$$

This is an extension of sect. 33 (1), and is found true on trial.

It will also be found on trial that

$$(a+b)^2 = a^2 + 2ab + b^2,\ (a-b)^2 = a^2 - 2ab + b^2;$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4;$$

from which we see that, when a binomial is raised to any power, (1) the indices of the first term decrease regularly, while those of the second increase regularly; (2) that the coefficient of the

second term is the same as the index of the power to which the binomial is raised; (3) that the number of terms is that index increased by 1; and (4) that the powers of $a-b$ differ from those of $a+b$ only in being negative where odd powers of the negative quantity, $-b$, occur. The coefficients for the 5th power are 1, 5, 10, 10, 5, 1; for the 6th, 1, 6, 15, 20, 15, 6, 1.

Hence, if we know any power of $a+b$, we can at once write the same power of any binomial by putting its first term for a, and its second for b or $-b$. Thus,

$$\because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$
$$\therefore (2x-3y)^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - 3y^3$$
$$= 8x^3 - 36x^2y + 54xy^2 - 27y^3.$$

If trinomials be bracketed into binomial form, their powers may also be found in this way. Thus,

$$\because a-b+c = (a-b)+c, \text{ or } a-(b-c),$$
$$\therefore (a-b+c)^3$$
$$= \{(a-b)+c\}^3 = (a-b)^3 + 3(a-b)^2c + 3(a-b)c^2 + c^3,$$
$$\text{or} = \{a-(b-c)\}^3 = a^3 - 3a^2(b-c) + 3a(b-c)^2 - (b-c)^3,$$

either of which can be easily written out in full.

Exercise XL.

Revise Ex. XV., 1-3, and find, by inspection, the squares of

1. $x+y,\ \ x-y,\ \ x+1,\ \ x-1,\ \ 1-x,\ \ x^2+y^2,\ \ x^2-y^2.$
2. $1+x^2,\ \ 1-x^2,\ \ x+3,\ \ x-4,\ \ 5-x,\ \ x+2y,\ \ x-3y.$
3. $3x-4y,\ \ ax-by,\ \ 2ab-3cd,\ \ 2a^2+3b^2,\ \ 3a^3-2b^2c,\ \ a^2b-xy^2.$
4. $x+y+z,\ \ x+y-z,\ \ x^2+x-1,\ \ 1-x-x^2.$
5. $x^2+2x-3,\ \ x^2+4x-5,\ \ x^2-3x+2,\ \ 2a-3b+4c.$
6. $x^2+y^2-z^2,\ \ 2x^2+3x-2,\ \ a+x+y+z,\ \ a-x+y-z.$

Find by inspection:

7. $(a+z)^3,\ \ (a-z)^3,\ \ (x+1)^3,\ \ (x-1)^3,\ \ (x+2)^3,\ \ (x-3)^3.$
8. $(x+y)^4,\ \ (x-y)^4,\ \ (1+x)^4,\ \ (1-x)^4,\ \ (a+x)^5,\ \ (a-x)^5.$
9. $(x+a)^6,\ \ (x-a)^6,\ \ (3a-b)^3,\ \ (x-2y)^3,\ \ (a+2b)^4.$
10. $(x+y+z)^3,\ \ (x-y-z)^3,\ \ (x^2-x-1)^3,\ \ (x^2-x+1)^3.$

EVOLUTION.

77. By *Evolution*, we mean *finding the roots of quantities.* **Sect 9.**

Where there is no bracket or vinculum, the root sign refers only to the number or letter to which it is prefixed.

Therefore, to indicate a root of a simple quantity having more than one factor, or of a compound quantity, these quantities should be bracketed or have a vinculum drawn over them.

Thus, $\sqrt{}9x^2 = 3x^2$, but $\sqrt{9x^2} = 3x$; $\sqrt{}9 + 16 = 3 + 16 = 19$, but $\sqrt{9 + 16} = \sqrt{25} = 5$.

78. 1. An *even* root of a *positive* quantity may be either *positive* or *negative*. Thus,

$\because a \times a = a^2$, and $-a \times -a = a^2$; $\therefore \sqrt{a^2} = a$ or $-a$, written $\pm a$.

2. No *negative* quantity can have an *even* root, for every quantity has all its even powers positive (sect. 74). Thus,

$$-a^2 = a \times -a, \text{ not } a \times a \text{ or } -a \times -a.$$

Such a quantity as $\sqrt{-a^2}$ is therefore called an *impossible quantity*.

3. An *odd* root of any quantity has *the same sign* as *the quantity itself.* Thus,

$a^3 = a \,.\, a \,.\, a$, $\therefore \sqrt[3]{a^3} = a$; $-a^3 = -a \,.\, -a \,.\, -a$, $\therefore \sqrt[3]{-a^3} = -a$.

79. 1. To find the Root of a Simple Quantity. Divide the index of each of its factors by the root index, and prefix the proper sign. Thus, because

$a^4 = a^2 \,.\, a^2$, $\therefore \sqrt[2]{a^4} = a^2 = a^{4 \div 2}$; $a^6 = a^2 \,.\, a^2 \,.\, a^2$, $\therefore \sqrt[3]{a^6} = a^2 = a^{6 \div 2}$

$8a^{12}b^{15} = 2^3 \,.\, a^4 \,.\, a^4 \,.\, a^4 \,.\, b^5 \,.\, b^5 \,.\, b^5$, $\therefore \sqrt[3]{8a^{12}b^{15}} = 2a^4b^5 = 2^{3 \div 3}a^{12 \div 3}b^{15 \div 3}$.

Similarly, $\sqrt{9a^6b^{10}} = \pm 3a^3b^5$; $\sqrt[4]{81x^8y^{12}} = \pm 3x^2y^3$; $\sqrt[5]{-32a^{10}x^{15}} = -2a^2x^3$.

But to divide one power by another, subtract the index of the divisor from that of the dividend (sect. 36). Thus, $a^6 \div a^2 = a^{6-2} = a^4$; but $\sqrt[2]{a^6} = a^{6 \div 2} = a^3$.

2. The *root of a fraction* is the root of its numerator divided by that of its denominator. Thus, since (sect. **62.** 1)

$$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}, \; \therefore \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{a}{b}. \text{ Similarly, } \sqrt[3]{\frac{a^3b^{12}}{8x^6y^9}} = \frac{ab^4}{2x^2y^3}.$$

3. Some roots cannot be found exactly; as the $\sqrt{2}$, $\sqrt{a^3}$, $\sqrt[3]{a^4}$, $\sqrt[4]{a^9}$. Such roots are called *surds* or *irrational quantities.*

80. TO FIND THE SQUARE ROOT OF A COMPOUND QUANTITY. 1. Arrange the terms of the given quantity according to powers of any one of its letters (sect. 39); 2. Find the square root of its first term, and set this down as the first term of the root; 3. Subtract the square of this term from the given quantity, and bring down the remainder; 4. Find how often twice this term is contained in the first part of the remainder, and set down the result as the next term of the root; 5. Take, as a divisor, (1) twice the former part of the root, and (2) the new part; 6. Multiply this divisor by the new part of the root, and subtract the result. If there is still a remainder, repeat the operations described in 4–6 till there be no remainder, or till the highest power of the leading letter in the remainder be lower than its highest power in the divisor.

Examples.

(1.)

$$\begin{array}{rl}
a) & a^2 + 2ab + b^2(a + b \\
 & a^2 \\
(a\times 2) + b & \\
=2a + b & \left|\begin{array}{l} 2ab + b^2 \\ 2ab + b^2 \end{array}\right.
\end{array}$$

(2.)

$$\begin{array}{rl}
a) & a^2 - 2ab + b^2(a - b \\
 & a^2 \\
(a\times 2) - b & \\
=2a - b & \left|\begin{array}{l} -2ab + b^2 \\ -2ab + b^2 \end{array}\right.
\end{array}$$

(3.)

$$\begin{array}{rl}
4x^2) & 16x^4 - 24x^3 + 25x^2 - 12x + 4(4x^2 - 3x + 2 \\
 & 16x^4 \\
(4x^2\times 2) - 3x & \\
=8x^2 - 3x & \left|\begin{array}{l} -24x^3 + 25x^2 \\ -24x^3 + 9x^2 \end{array}\right. \\
2(4x^2 - 3x) + 2 & \\
=8x^2 - 6x + 2 & \left|\begin{array}{l} 16x^2 - 12x + 4 \\ 16x^2 - 12x + 4 \end{array}\right.
\end{array}$$

The answer to (1) is $a + b$, or $-(a + b) = -a - b$ (52.1); to (2), $a - b$, or $-(a - b) = b - a$; to (3), $4x^2 - 3x + 2$, or $-(4x^2 - 3x + 2) = -4x^2 + 3x - 2$.

In practice it is sufficient, as in arithmetic, to double the last part of each divisor and annex the new part. Thus, in (3), first $4x^2$ is doubled and $-3x$ annexed; then $-3x$ is doubled and $+2$ annexed.

81. TO FIND THE CUBE ROOT OF A COMPOUND QUANTITY. 1. Arrange its terms as in sect. 80; 2. Find the cube root of its

first term, and set this down as the first term of the root; 3. Subtract the cube of this term from the given quantity, and bring down the remainder; 4. Find how often 3 times the square of this term is contained in the first part of the remainder, and set down the result as the next term of the root; 5. Take, as a divisor, (1) three times the square of the former part of the root, (2) three times the product of the former part and the new part, and (3) the square of the new part; 6. Multiply this divisor by the new part of the root, and subtract the result. If there is still a remainder, repeat the operations described in 4–6, till there be no remainder, or till the highest power of the leading letter in the remainder be lower than its highest power in the divisor.

Examples.

(1.)

$$\begin{array}{rl|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & (a + b \\
 & a^3 & \\
3a^2 + 3(a \times b) + b^2 & & 3a^2b + 3ab^2 + b^3 \\
= 3a^2 + 3ab + b^2 & & 3a^2b + 3ab^2 + b^3
\end{array}$$

(2.)

$$\begin{array}{rr|l}
 & x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1 & (x^2 - x + 1 \\
 & x^6 & \\
3x^4 + 3(x^2 \cdot - x) + x^2 & & -3x^5 + 6x^4 - 7x^3 \\
 & = 3x^4 - 3x^3 + x^2 & -3x^5 + 3x^4 - x^3 \\
3(x^2 - x)^2 & = 3x^4 - 6x^3 + 3x^2 & 3x^4 - 6x^3 + 6x^2 - 3x + 1 \\
3(x^2 - x) \times 1 = & 3x^2 - 3x & \\
1^2 = & 1 & \\
 & 3x^4 - 6x^3 + 6x^2 & 3x^4 - 6x^3 + 6x^2 - 3x + 1 \\
 & -3x + 1 &
\end{array}$$

82. The 4th root of a quantity is the *square root* of its *square root;* its 6th root, the *square root* of its *cube root,* or the *cube root* of its *square root;* its 8th root, the *square root* of its *fourth root;* and so on. For $a^4 = (a^2)^2$; $a^6 = (a^2)^3$ or $(a^3)^2 \cdot a^8 = (a^2)^4$; &c. (Sect. 75.)

EXERCISE XLI.

Find

1. $\sqrt{a^2}$, $\sqrt{x^4}$, $\sqrt{y^6}$, $\sqrt{a^2b^2}$, $\sqrt{a^2b^2}$, $\sqrt{4a^4}$, $\sqrt{4a^4}$; $\sqrt{64x^{12}}$.

2. $\sqrt{9}x^2y^4$, $\sqrt{9x^2}y^4$, $\sqrt{9x^2y^4}$, $\sqrt{16}x^4y^6z^8$, $\sqrt{16x^4}y^6z^8$, $\sqrt{16x^4y^6z^8}$.

3. $\sqrt{4}x^6 \times 9y^4$, $\sqrt{4x^6} \times 9y^4$, $\sqrt{4x^6 \times 9y^4}$, $\sqrt{25a^8 \times 49b^{16}}$.

4. $\sqrt{a^{12}}$, $\sqrt[3]{a^{12}}$, $\sqrt[4]{a^{12}}$, $\sqrt[6]{a^{12}}$, $\sqrt[3]{x^3}y^6$, $\sqrt[3]{x^3y^6}$, $\sqrt[3]{-x^6}$.

5. $\sqrt[3]{8}x^3$, $\sqrt[3]{8x^3}$, $\sqrt[3]{-8x^3}$, $\sqrt[4]{16a^4b^8}$, $\sqrt[4]{81x^8y^{12}}$, $\sqrt[3]{-27x^9}$.

6. $\sqrt[5]{x^5}$, $\sqrt[5]{-x^{10}}$, $\sqrt[5]{32}a^{10}$, $\sqrt[5]{32a^{15}}$, $\sqrt[5]{-32a^{10}}$, $\sqrt[6]{64a^6x^{18}}$.

7. $\sqrt{\frac{a^2}{x^4}}$, $\sqrt[3]{\frac{x^3}{y^6}}$, $\sqrt[3]{-\frac{a^3}{x^9}}$, $\sqrt{\frac{25a^2b^4}{36a^6y^8}}$, $\sqrt[4]{\frac{16x^4}{81y^8}}$, $\sqrt[5]{-\frac{a^{10}b^{15}}{32x^5}}$.

Find the square roots of

8. $a^2+2ax+x^2$, $x^2-2xy+y^2$, x^2+6x+9, $4x^2-12x+9$.

9. $4a^2+12ab+9b^2$, $9a^2+24ab+16b^2$, $25a^2-60ab+36b^2$.

10. $a^2+2ab+b^2+2ac+2bc+c^2$, $x^2-2xy+y^2+2xz-2yz+z^2$.

11. $x^4-2x^3+3x^2-2x+1$, $x^6+2x^5+3x^4-x^2-2x+1$.

12. $1-2x-x^2+2x^3+x^4$, $9x^4-12x^3+10x^2-4x+1$.

13. $x^4+4x^3-2x^2-12x+9$, $4x^4-12x^3+25x^2-24x+16$.

14. $1-6x+13x^2-12x^3+4x^4$, $x^6-6x^5+13x^4-12x^3+4x^2$.

15. $x^2+4xy+4y^2+6xz+12yz+9z^2$, $9a^2-24ab+16b^2+30ac-40bc+25c^2$, $4a^4-12a^3b-7a^2b^2+24ab^3+16b^4$.

16. $x^6-4x^5+10x^3-4x+1$, $a^6+4a^5x-10a^3x^3+4ax^5+x^6$.

Find the cube roots of

17. $x^3-3x^2y+3xy^2-y^3$, $a^3+9a^2b+27ab^2+27b^3$.

18. $27a^3+54a^2b+36ab^2+8b^3$, $64a^3-144a^2x+108ax^2-27x^3$.

19. $a^3-3a^2b+3ab^2-b^3+3a^2c-6abc+3b^2c+3ac^2-3bc^2+c^3$.

20. $8-12x+18x^2-13x^3+9x^4-3x^5+x^6$, $x^6-6x^5+3x^4+28x^3-9x^2-54x-27$, $8a^6-48a^5x+132a^4x^2-208a^3x^3+198a^2x^4-108ax^5+27x^6$, $x^6+3x^5-5x^3+3x-1$.

21. Find the 4th roots of $16x^4-32x^3+24x^2-8x+1$, $81x^4+432x^3+864x^2+768x+256$, $81a^4-216a^3b+216a^2b^2-96ab^3+16b^4$, $256x^4-1280x^3+2400x^2-2000x+625$.

22. Find the 6th roots of $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$, $x^6+18x^5+135x^4+540x^3+1215x^2+1458x+729$.

INDICES.

83. In Part I. of *The Standard Algebra* we read: 'A small figure placed over a quantity, and to its right, indicates the power to which it is raised, or the number of factors of which it is composed, and is called the *index* or *exponent* of the power.' *

Such a definition is necessary and suitable for the earlier parts of the subject, but it is altogether inadequate to express the operations which we shall now proceed to consider. Thus, in

$$a^4, \quad a^{-3}, \quad a^{\frac{1}{2}}, \quad a^{-\frac{1}{3}},$$

4, -3, $\frac{1}{2}$, and $-\frac{1}{3}$, are all *indices;* but it would scarcely be intelligible to say that -3, $\frac{1}{2}$, and $-\frac{1}{3}$, denote the *number of factors* which must be multiplied together to produce the required power.

We shall therefore determine the meaning of such expressions; and shall simply consider an index to be *a convention (or convenient form) for expressing some operation indicated by its form and character.*

A consideration of the subject must necessarily be confined to those parts of Algebra which affect the form and condition of the *indices;* and therefore we must determine the application of our convention to the rules of Multiplication, Division, Involution, and Evolution.

I. We know that $a^2 = a \times a$, that $a^3 = a \times a \times a$, and that $a^2 \times a^3 = \overline{a \times a} \times \overline{a \times a \times a} = a^5$. And so we have, generally,

$$a^m = a \times a \times a \times \&c. \ldots \text{ to } m \text{ factors,}$$
$$a^n = a \times a \times a \times \&c. \ldots \text{ to } n \text{ factors,}$$
$$\text{and } a^m \times a^n = (a \times a \times a \times \&c. \ldots \text{ to } m \text{ factors})$$
$$\times (a \times a \times a \times \&c. \ldots \text{ to } n \text{ factors}),$$
$$= a \times a \times a \times \&c. \ldots \text{ to } \overline{m+n} \text{ factors,}$$
$$= a^{m+n},$$

the index, $\overline{m+n}$, denoting, of course, when m and n are *integral* and *positive*, the number of factors which are multiplied together to form the power.

II. But let us consider the conditions when a *negative* exponent appears in the operation. From our general form, we have

* Page 9, section 8.

$$a^m \times a^n \times a^p = a^{m+n+p},$$
$$\text{and } a^m \times a^n \times a^p \times a^{-p} = a^{m+n+p-p}$$
$$= a^{m+n}.$$

$$\text{But } a^m \times a^n \times a^p \times \frac{1}{a^p} \text{ also equals } a^{m+n}.$$

$$\therefore a^m \times a^n \times a^p \times a^{-p} = a^m \times a^n \times a^p \times \frac{1}{a^p},$$

$$\text{and } \therefore a^{-p} = \frac{1}{a^p}.$$

We therefore say that a^{-p} is the *reciprocal* of a^p, but this is a term which requires explanation. *When the product of any two quantities is* UNITY, *one of them is said to be the* RECIPROCAL *of the other*. Thus,

$$\frac{b}{a} \times \frac{a}{b} = 1;$$

$$\therefore \frac{a}{b} \text{ is the } \textit{reciprocal} \text{ of } \frac{b}{a}.$$

We have now determined the precise meaning of our convention, a^{-p}.

III. Again, what operation is represented by $a^{\frac{1}{q}}$?

In order to make *fractional* indices comply with the requirements of our general form, $a^m \times a^n = a^{m+n}$, we must consider

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} \text{ to equal } a^1, \text{ and}$$
$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} \text{ to equal } a^1 \text{ too;}$$

and so on. But, if *two* like quantities are multiplied together, each of them is the *square root* of the product; and if *three* like quantities are multiplied together, each of them is the *cube root* of the product. Therefore,

$$a^{\frac{1}{2}} = \sqrt{a}, \text{ and } a^{\frac{1}{3}} = \sqrt[3]{a};$$

where we see that *a fractional index denotes the extraction of the root which is represented by the denominator.*

So, too, if $a^{\frac{1}{q}}$ be multiplied by $a^{\frac{1}{q}}$, *to q factors*, we have

$$a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \text{\&c.} \ldots \text{ to } q \text{ factors}$$
$$= a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\text{\&c.} \ldots \text{ to } q \text{ terms}}$$

$$= a^{(\frac{1}{q})q} = a^{\frac{q}{q}} = a^1.$$

Therefore, $a^{\frac{1}{q}} = \sqrt[q]{a}$ (the q^{th} root of a), as before.

IV. Having now considered the most important cases, we will proceed to determine a meaning for $a^{\frac{p}{q}}$.

If q factors of $a^{\frac{p}{q}}$ be multiplied together, we shall have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \&c \ \ldots \text{ to } q \text{ factors}$$
$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\,\&c.\ \ldots\ \text{to } q \text{ terms}}$$
$$= a^{\left(\frac{p}{q}\right)q} = a^{\frac{pq}{q}} = a^{p}.$$

But, if a product is produced by the multiplication of q like factors, one of the factors is the q^{th} *root* of the product or power. Therefore, $a^{\frac{p}{q}} =$ *the* q^{th} *root* of the p^{th} *power of* a. It will also be observed, from the last two considerations, that the *numerator* of a fractional index denotes a *power*, while the *denominator* denotes a *root*.

V. It is also necessary to ascertain the precise significance of a^0, and this we will now do. We know that

$$a^m \times a^n \times a^p \times a^0 = a^{m+n+p+0}$$
$$= a^{m+n+p}.$$

But $\quad a^m \times a^n \times a^p \times 1 = a^{m+n+p}.$

$\therefore \quad a^m \times a^n \times a^p \times a^0 = a^m \times a^n \times a^p \times 1.$

$$\therefore a^0 = 1.$$

84. There are other demonstrations which need consideration, in order to establish completely the various formulæ and conventions which a more extended study of the subject must necessarily include. But the foregoing are the most important; and, for the purposes of this work, are amply sufficient. The others, too, are so easily derivable from those already given, that a simple knowledge of fractional combinations will enable the student to manipulate, with confidence, the various examples which may come under his notice.

Examples.

(1) $a^7 \times b^6 \times a^8 \times b^3 \times c^2 = a^{7+8}\, b^{6+3}\, c^2$
$$= a^{15}\, b^9\, c^2.$$

(2) $m^3 \times n^6 \times p^{-4} \times m^{-5} \times p^{12} \times n^{-7}$
$$= m^{3-5}\, n^{6-7}\, p^{12-4} = m^{-2}\, n^{-1}\, p^8$$
$$= \frac{p^8}{m^2\, n}.$$

(3) $a^{\frac{1}{2}} \times c^{-\frac{2}{3}} \times b^{\frac{1}{4}} \times a^{-\frac{1}{6}} \times b^{-\frac{1}{8}} \times c^{\frac{1}{3}}$
$$= a^{\frac{1}{2}-\frac{1}{6}}\, b^{\frac{1}{4}-\frac{1}{8}}\, c^{\frac{1}{3}-\frac{2}{3}} = a^{\frac{1}{3}}\, b^{\frac{1}{8}}\, c^{-\frac{1}{3}}$$
$$= \frac{a^{\frac{1}{3}}\, b^{\frac{1}{8}}}{c^{\frac{1}{3}}}.$$

(4) $a^{2n} + b^{-n} + c^{\frac{n}{2}}$

$a^{2n} + b^{n} - c^{-\frac{n}{2}}$

$a^{4n} + a^{2n}b^{-n} + a^{2n}c^{\frac{n}{2}}$

$a^{2n}b^{n} + b^{0} + b^{n}c^{\frac{n}{2}}$

$- a^{2n}c^{-\frac{n}{2}} - b^{-n}c^{-\frac{n}{2}} - c^{0}$

$$a^{4n} + a^{2n}b^{-n} + a^{2n}b^{n} + a^{2n}c^{\frac{n}{2}} - a^{2n}c^{-\frac{n}{2}} + b^{n}c^{\frac{n}{2}} - b^{-n}c^{-\frac{n}{2}} + 1 - 1$$
$$= a^{4n} + a^{2n}(b^{-n} + b^{n} + c^{\frac{n}{2}} - c^{-\frac{n}{2}}) + b^{n}c^{\frac{n}{2}} - b^{-n}c^{-\frac{n}{2}}.$$

(5) $p^{-\frac{1}{2}}q + q^{\frac{1}{2}} + p^{\frac{1}{2}}\,)\,p^{-\frac{1}{2}}q^{\frac{3}{2}} + p^{\frac{1}{2}}q^{\frac{1}{2}} + p^{\frac{3}{2}}q^{-\frac{1}{2}}\,(\,q^{\frac{1}{2}} - p^{\frac{1}{2}} + pq^{-\frac{1}{2}}$

$p^{-\frac{1}{2}}q^{\frac{3}{2}} + q + p^{\frac{1}{2}}q^{\frac{1}{2}}$

$- q + p^{\frac{3}{2}}q^{-\frac{1}{2}}$

$- q - p^{\frac{1}{2}}q^{\frac{1}{2}} - p$

$p^{\frac{1}{2}}q^{\frac{1}{2}} + p + p^{\frac{3}{2}}q^{-\frac{1}{2}}$

$p^{\frac{1}{2}}q^{\frac{1}{2}} + p + p^{\frac{3}{2}}q^{-\frac{1}{2}}$

$\cdot \quad \cdot \quad \cdot \quad \cdot$

Note.—It is important, in Division, to arrange the terms similarly in the dividend and divisor—that is, in the same order of indices; and to observe the same arrangement in the ***first terms*** of the quantities brought down. Thus, in example (5), we have in

Dividend:	$p^{-\frac{1}{2}}$,	$p^{\frac{1}{2}}$,	$p^{\frac{3}{2}}$
Divisor:	$p^{-\frac{1}{2}}$,	p^{0},	$p^{\frac{1}{2}}$
First Terms:	$p^{-\frac{1}{2}}$,	p^{0},	$p^{\frac{1}{2}}$.

Exercise XLII.

If $x = 16$, $y = 36$, $z = 64$, and $q = 8$, find the values of

(1) x^{-2}. (2) $x^{\frac{1}{2}}y^{-\frac{1}{2}}$.

(3) $\dfrac{x^{-\frac{1}{4}}}{y^{-\frac{1}{2}}z^{-\frac{1}{3}}}$. (4) $x^{-\frac{1}{4}} + y^{-\frac{1}{2}} - z^{-\frac{1}{6}}$.

(5) $(40x + 10y)^{-\frac{1}{3}}$.

(6) $\sqrt[3]{q} + \sqrt[4]{x^{2}} - \dfrac{1}{(32)^{-\frac{1}{5}}}$.

(7) $(z^{-\frac{1}{2}})^{-2}$. (8) $\sqrt{\{(z^{-\frac{1}{6}})^{2}\}}$.

(9) $x^{\frac{1}{3}} + x^{\frac{1}{4}} - y^{\frac{1}{2}} + (4y)^{\frac{1}{2}} + z^{\frac{1}{3}} - \left(\frac{z}{8}\right)^{\frac{1}{3}} - \left(\frac{8}{z}\right)^{\frac{1}{3}} - (128)^{-\frac{1}{7}}$.

(10) $\left(z^{\frac{1}{3}} \times z^{-\frac{1}{3}} \times \frac{1}{z^{\frac{1}{3}}} \times z^{-\frac{1}{3}} \times \frac{1}{z^{-\frac{1}{3}}}\right) - z^{\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}+\frac{1}{3}}$.

(11) $\sqrt[q]{x^2} - x^{\frac{2}{q}} + \sqrt[4]{2^q} - 2^{\frac{q}{4}} - y^{-\frac{q}{s}} + \frac{1}{(xyz)^{-\frac{1}{3}}}$.

Multiply

(12) $m^{3x} + m^{\frac{3x}{2}}n^{\frac{x}{2}} + n^{3x}$ by $m^{3x} - m^{\frac{3x}{2}}n^{\frac{3x}{2}} + n^{3x}$.

(13) $a^{\frac{1}{3}} + b^{\frac{1}{3}}$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

(14) $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$ by $\sqrt{x^2} + \sqrt{y^2}$.

(15) $m^{-4} - m^{-3} + m^{-2} - m^{-1} + 1$ by $m^{-1} - 1$.

(16) $m^{-4} + m^{-3} + m^{-2} + m^{-1} + 1$ by $m^{-1} - 1$.

(17) $\frac{5}{2}a^m + \frac{3}{2}b^n - \frac{1}{2}c^p$ by $\frac{5}{2}a^{2m-m} + \frac{3}{2}b^{2n-n} + \frac{1}{2}c^{2p-p}$.

(18) $a^{\frac{4}{3}}b^{\frac{1}{3}} + a^{\frac{2}{3}}b^{-\frac{1}{3}} + ab^{-\frac{4}{3}}$ by $a^{\frac{2}{3}}b^{\frac{2}{3}} - a^{-1}b^{\frac{2}{3}}$.

(19) $m^{-2z+1} - m^{-z+\frac{1}{2}}n^{-z+\frac{1}{2}} + n^{-2z+1}$ by $m^{-z+\frac{1}{2}} + n^{-z+\frac{1}{2}}$.

(20) $A^{\frac{m+n}{2}} + A^{\frac{m+n}{4}}B^{\frac{m+n}{4}} + B^{\frac{m+n}{2}}$ by $A^{\frac{m+n}{2}} - A^{\frac{m+n}{4}}B^{\frac{m+n}{4}} + B^{\frac{m+n}{2}}$.

Divide

(21) $a^{\frac{4}{3}} - b^{\frac{4}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

(22) $16p^5q^{-3} - 40p^4q^{-2} + 56p^3q^{-1} - 40p^2 + 16pq$ by the product of $2pq^{-1}$ and $p^2q^{-2} - pq^{-1} + 1$.

(23) $m^{\frac{5}{2}} - m^{\frac{3}{2}}n + mn^{\frac{3}{2}} - 2m^{\frac{1}{2}}n^2 + n^{\frac{5}{2}}$ by $(m + n)(m^{\frac{1}{2}} - n^{\frac{1}{2}})$.

(24) $mn^{-1} + m^{\frac{5}{6}}n^{-\frac{4}{5}} + m^{\frac{2}{6}}n^{-\frac{2}{5}} - m^{-\frac{2}{5}}n^{\frac{2}{5}} - m^{-\frac{1}{3}}n^{-\frac{1}{5}} - m^{-1}n$ by $m^{\frac{4}{5}}n^{-\frac{1}{5}} + m^{\frac{1}{5}}n^{-\frac{2}{5}} + m^{-\frac{1}{5}}n^{\frac{4}{5}}$.

(25) $m^{\frac{3x}{2}} - m^{-\frac{3x}{2}}$ by $m^{\frac{x}{2}} - m^{-\frac{x}{2}}$.

(26) $2x^{2p}y^{-\frac{16}{5}p} - 17x^{\frac{12}{5}p}y^{-\frac{8}{5}p} + 5x^{\frac{14}{5}p} + 24x^{\frac{16}{5}p}y^{\frac{8}{5}p}$ by $x^{\frac{4}{5}p}y^{-\frac{12}{5}p} - 7x^{\frac{6}{5}p}y^{-\frac{2}{5}p} - 8x^{\frac{8}{5}p}y^{\frac{4}{5}p}$.

(27) $9x^{-3} - 20x^{-\frac{5}{2}}y^{-\frac{1}{2}} + \frac{716}{45}x^{-2}y^{-1} - \frac{16}{3}x^{-\frac{3}{2}}y^{-\frac{3}{2}} + \frac{16}{25}x^{-1}y^{-2}$ by

$3x^{-\frac{3}{2}} - \frac{10}{3}x^{-1}y^{-\frac{1}{2}} + \frac{4}{5}x^{-\frac{1}{2}}y^{-1}$.

Expand, as required, the following expressions:

(28) $\left(\frac{1}{2}a^{\frac{1}{2}} - \frac{1}{3}b^{\frac{1}{3}}\right)^2$.

(29) $(a^{-m} + 2a^{-\frac{m}{2}} - 3a^{-\frac{m}{3}})^2$.

(30) $\left(\frac{2}{7}pq^{-1} + \frac{3}{4}p^{-1}q - 5\right)^2$.

(31) $(a^{\frac{1}{4}} - b^{\frac{1}{3}})^3$.

(32) $(2a^{-\frac{1}{2}} - 2a + 3a^{\frac{1}{2}})^3$.

(33) $(2m^{-\frac{2}{3}} - 3n^{-\frac{3}{4}})^4$.

(34) Find the fourth term of $(2^{\frac{1}{2}} + 3^{-\frac{1}{3}})^6$.

Extract the square roots of

(35) $\frac{1}{64}m^{-2p} + \frac{1}{36}m^{-\frac{3p}{2}} + \frac{1}{81}m^{-p}$.

(36) $\frac{1}{16}x^{-\frac{3}{2}} - \frac{1}{10}x^{-\frac{3}{4}}y^{\frac{3}{4}} + \frac{1}{25}y^{\frac{3}{2}}$.

(37) $9m^{-4p} - 24m^{-3p} - 30m^{-\frac{5p}{2}} + 16m^{-2p} + 40m^{-\frac{3p}{2}} + 25m^{-p}$.

(38) $49m^{-5} - 112m^{-\frac{9}{2}}n^2 + 64m^{-4}n^4 + 126m^{-\frac{5}{2}}n^{\frac{5}{2}} - 144m^{-2}n^{\frac{9}{2}}$
$+ 81n^5$.

Obtain the cube roots of

(39) $216x^{-\frac{3m}{2}} - 756x^{-m}y^{-\frac{m}{3}} + 882x^{-\frac{m}{2}}y^{-\frac{2m}{3}} - 343y^{-m}$.

(40) $64m^{6p-\frac{3}{4}} - 528m^{\frac{19}{4}p-\frac{5}{2}} + 1452m^{\frac{7}{2}p-\frac{17}{4}} - 1331m^{\frac{9}{4}p-6}$.

(41) $8a^{\frac{3}{4}}b^{-\frac{3}{4}} - 12a^{\frac{1}{2}}b^{-\frac{1}{2}} + 42a^{\frac{1}{4}}b^{-\frac{1}{4}} - 37 + 63a^{-\frac{1}{4}}b^{\frac{1}{4}} - 27a^{-\frac{1}{2}}b^{\frac{1}{2}}$
$+ 27a^{-\frac{3}{4}}b^{\frac{3}{4}}$.

(42) Find the fourth root of $81a^6b^{-6} - 432a^{\frac{21}{4}}b^{-\frac{21}{4}} + 864a^{\frac{9}{2}}b^{-\frac{9}{2}}$
$- 768a^{\frac{15}{4}}b^{-\frac{15}{4}} + 256a^3b^{-3}$.

Reduce the following fractions to lowest terms, by resolution into factors:

(43) $\dfrac{p + p^{\frac{3}{2}} + p^2 + p^{\frac{5}{2}}}{2p^2 + 2p^{\frac{5}{2}} + 3p^3 + 3p^{\frac{7}{2}}}$.

(44) $\dfrac{a^{\frac{1}{3}}p^{-2q} + 3a^{\frac{1}{2}}p^{-q} - b^{\frac{1}{3}}p^{-2q} - 3b^{\frac{1}{2}}p^{-q}}{a^{\frac{1}{3}}p^{-2q} - 3a^{\frac{1}{2}}p^{-q} - b^{\frac{1}{3}}p^{-2q} + 3b^{\frac{1}{2}}p^{-q}}$.

(45) $\dfrac{x^{\frac{5}{2}} - 14x^{\frac{3}{2}} + x^{-\frac{3}{2}} - x^2y^{\frac{1}{2}} + 14xy^{\frac{1}{2}} - x^{-2}y}{x^3 - 14x^2 + x^{-1} - x^2y + 14xy - x^{-2}y}$.

SURDS.

85. It is but a short step from *indices* to *surds*—a surd being an *irrational* quantity whose root cannot be accurately determined. Therefore,

$$\sqrt{7}, \quad 9^{\frac{1}{3}}, \quad \sqrt[4]{10}, \quad (\tfrac{7}{8})^{\frac{1}{2}},$$

are all surds, the required root of each being an irrational quantity. The *even* roots of negative quantities, as $\sqrt{-3}$, $(-5)^{\frac{1}{2}}$, are purely *imaginary* or *impossible* quantities, and not, therefore, really surds at all.

86. It will somewhat lessen the first uncertainties connected with the manipulation of surds, if the student bear carefully in mind that the *form* of the expression adds no element of difficulty. Whatever he has been accustomed to do with x, he can do with $\sqrt{x}$; and just as

$$5x + 3x - 2x = 6x,$$
$$5\sqrt{x} + 3\sqrt{x} - 2\sqrt{x} = 6\sqrt{x}.$$

The beginner, however, often thinks that the processes for the manipulation of surds, are much more complicated and difficult than ordinary algebraic processes, and gets into difficulties in consequence of the erroneous supposition.

87. The following rules, proofs, and observations, must be thoroughly understood and carefully remembered.

I. A *rational* quantity may be expressed in the *form* of a surd, by raising the given quantity to the power which is indicated by the *denominator* of the surd-index. Thus,

$$\sqrt{256}, \quad \sqrt[3]{64}, \quad (\tfrac{1}{8})^{\frac{1}{3}}, \quad \sqrt[5]{-32},$$

are all rational quantities in the form of surds; since

$$\sqrt{256} = 16, \quad \sqrt[3]{64} = 4, \quad (\tfrac{1}{8})^{\frac{1}{3}} = \tfrac{1}{2}, \quad \text{and } (-32)^{\frac{1}{5}} = -2.$$

In the same manner, if x, y, z, be rational quantities,

$$\sqrt{x^2}, \quad \sqrt[3]{y^3}, \quad (z^4)^{\frac{1}{4}},$$

are also rational quantities in the form of surds.

II. Surds may be *added*, *subtracted*, *multiplied*, and *divided*, like ordinary rational quantities. Thus,

$$(\alpha)\ 4\sqrt{a+x} + \tfrac{2}{3}\sqrt{a+x} - \tfrac{14}{3}\sqrt{a+x} = \tfrac{11}{3}\sqrt{a+x};$$

$$(\beta)\ \sqrt{x-y} \times 2\sqrt{x+y} \times \tfrac{1}{2}\sqrt{x-y} \div \sqrt{x+y}$$
$$= \frac{\frac{1}{2}(x-y) \times 2\sqrt{x+y}}{\sqrt{x+y}} = x - y.$$

III. It will be evident from I. and II., that the *co-efficient* of a surd may be put under the radical sign, and be multiplied by the surd-factor. Thus,

$$6\sqrt{7} = \sqrt{6 \times 6 \times 7} = \sqrt{252};\ (a+b)\sqrt{c+d} = \sqrt{(a+b)^2(c+d)};$$
$$(x+y)\sqrt{x+y} = \sqrt{(x+y)^2(x+y)} = \sqrt{(x+y)^3}.$$

IV. It follows, therefore, that in a surd expression, if there be a factor whose *root* can be extracted—that is, the root corresponding to the one which is expressed by the surd-index—that root can appear as the *co-efficient* of the remaining surd-factor. Thus,

(1) $\sqrt{343} = \sqrt{7 \times 7 \times 7} = 7\sqrt{7}$;

(2) $\sqrt[3]{11979} = \sqrt[3]{11 \times 11 \times 11 \times 9} = 11\sqrt[3]{9}$;

(3) $\sqrt[4]{a^5 - a^4 x} = \sqrt[4]{a^4(a-x)} = a(a-x)^{\frac{1}{4}}$.

V. When the quantity under the radical sign is as *small* as possible, but still *integral*, the surd is said to be in its *simplest form*. Thus,

(1) $\sqrt{240}$ equals, in its simplest form, $4\sqrt{15}$;

(2) $\sqrt{(a+b)^3}$ " " " $(a+b)\sqrt{a+b}$;

(3) $\sqrt{\tfrac{4}{75}}$ " " " $\tfrac{2}{15}\sqrt{3}$.

N.B. $\sqrt{\tfrac{4}{75}} = 2\sqrt{\tfrac{1}{5 \times 5 \times 3}} = \tfrac{2}{5}\sqrt{\tfrac{1}{3}} = \tfrac{2}{5}\sqrt{\tfrac{3}{3 \times 3}} = \tfrac{2}{15}\sqrt{3}$.

VI. (α) *Similar* surds are those which have, or may be made to have, the *same* quantity under the surd-index. Thus,

$$\sqrt{8},\quad \sqrt{18},\quad \sqrt{200},\quad \sqrt{\tfrac{4}{50}},$$

are all similar surds, because they equal, respectively,

$$2\sqrt{2},\quad 3\sqrt{2},\quad 10\sqrt{2},\quad \tfrac{1}{5}\sqrt{2},$$

the surd-factor being the same throughout.

(β) *Dissimilar* surds are those which have not, and cannot be made to have, the *same* quantity under the surd-index. Thus,

$$6\sqrt{2},\quad 7\sqrt{3},\quad 5\sqrt{5},\quad \sqrt{7},$$

are all dissimilar surds.

Note.—Of course, in the addition and subtraction of *dissimilar* surds, they can only be connected by their respective signs. Just as we say,

$$a + 2a - b - 3c + 2b = 3a + b - 3c,$$

so $\sqrt{2} + 2\sqrt{2} - \sqrt{3} - 3\sqrt{5} + 2\sqrt{3} = 3\sqrt{2} + \sqrt{3} - 3\sqrt{5}$.

VII. Surds of the *same order* are those whose INDICES, or surd-symbols, denote that the *same root* has to be extracted in each expression. Thus,

$$11\sqrt{a}, \quad (x + y)\sqrt{a + b}, \quad (m + n)q^{\frac{1}{2}},$$

are all surds of the same order; but

$$11\sqrt[3]{a}, \quad (x + y)\sqrt[3]{a + b}, \quad (m + n)q^{\frac{1}{3}},$$

are surds of *different* orders.

Note.—When the *indices* are the same, and the *surd-factors* under them alike too, the surds are not only of the same order, but are also called *like* surds. Thus,

$$11\sqrt{a + b}, \quad (x + y)\sqrt{a + b}, \quad (m + n)(a + b)^{\frac{1}{2}},$$

are like surds.

VIII. Surds of *different* orders may be transformed into surds of the *same* order, by reducing their indices to the same denomination. Thus,

$$a^{\frac{1}{2}}, \quad (m + n)^{\frac{2}{3}}, \quad (\tfrac{x}{8})^{\frac{3}{4}}, \quad (5y)^{\frac{5}{6}},$$

which are surds of different orders, may be reduced to

$$a^{\frac{6}{12}}, \quad (m + n)^{\frac{8}{12}}, \quad (\tfrac{x}{8})^{\frac{9}{12}}, \quad (5y)^{\frac{10}{12}},$$

which are surds of the same order, because, in each case, the same root (the 12th) has to be extracted.

IX. It is often necessary to ascertain the *relative magnitudes* of surd expressions. In order to do this:

(1) Reduce the indices to the same denomination;

(2) Convert the expressions into *entire* surds; that is, introduce the co-efficient under the radical signs; and

(3) Raise the quantities to the powers expressed by their respective numerators.

As the same root has now to be extracted in each case, it is clear that the relative magnitudes of the surd expressions will be at once apparent. Thus,

$$3\sqrt[3]{4}, \ 4\sqrt[2]{3}, \ 2\sqrt[6]{1000},$$

$$= \sqrt[3]{3^3 \times 4}, \ \sqrt{4^2 \times 3}, \ \sqrt[6]{2^6 \times 1000},$$

$$= (108)^{\frac{1}{3}},\ (48)^{\frac{1}{2}},\ (64000)^{\frac{1}{6}},$$
$$= (108)^{\frac{2}{6}},\ (48)^{\frac{3}{6}},\ (64000)^{\frac{1}{6}},$$
$$= (11664)^{\frac{1}{6}},\ (110592)^{\frac{1}{6}},\ (64000)^{\frac{1}{6}},$$

from which we can easily discover the proportional magnitudes.

X. Fractions whose *denominators* are compound quantities, containing *quadratic* surds, are converted into equivalent fractions in their simplest forms, by *rationalising* their denominators. The well-known formula, $(a - b)(a + b) = a^2 - b^2$, enables us to effect this reduction easily, as $a^2 - b^2$ must be rational, if *a or b*, or *a and b* are quadratic surds. Thus,

$$(1)\quad \frac{16}{\sqrt{5}+\sqrt{3}} = \frac{16(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})} = \frac{16(\sqrt{5}-\sqrt{3})}{5-3}$$
$$= 8(\sqrt{5}-\sqrt{3});$$

$$(2)\quad \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})} = \frac{a+b+2\sqrt{ab}}{a-b}.$$

XI. There are other important propositions which may be considered and proved, but no more are needed for the purposes of *The Standard Algebra.* The student, however, should bear in mind that,

In any *equation*, which has rational and irrational quantities *on each side*, the rational and irrational parts are respectively *equal to one another.*

88. We will now demonstrate a method, based on the assumption in the preceding article, for extracting the *square* root of a *binomial* expression, one of whose terms is a *quadratic* surd. Suppose the square root of $60 + 14\sqrt{11}$ to be the quantity required.

$$\text{Let}\quad \sqrt{x} + \sqrt{y} = \sqrt{60 + 14\sqrt{11}};$$
$$\text{Then}\quad (\sqrt{x} + \sqrt{y})^2 = 60 + 14\sqrt{11};$$

From which

$$x + y = 60\ (\textit{rational}\text{ parts}),$$

and
$$2\sqrt{xy} = 14\sqrt{11}\ (\textit{irrational}\text{ parts}).$$

The solution of these equations gives

$$x = 49 \text{ or } 11, \text{ and } y = 11 \text{ or } 49.$$

$\therefore \sqrt{x} + \sqrt{y}$ (that is, the square root of $60 + 14\sqrt{11}$) $= \sqrt{49} + \sqrt{11} = 7 + \sqrt{11}$.

Examples.

(1) Simplify $\sqrt{80} - \sqrt{1445} + \sqrt{605} + (33\tfrac{4}{5})^{\frac{1}{2}}$.

$$\begin{aligned}
&\sqrt{80} - \sqrt{1445} + \sqrt{605} + \sqrt{\tfrac{169}{5}} \\
&= \sqrt{16 \times 5} - \sqrt{289 \times 5} + \sqrt{121 \times 5} + 13\sqrt{\tfrac{1}{5}} \\
&= 4\sqrt{5} - 17\sqrt{5} + 11\sqrt{5} + 13\sqrt{\tfrac{5}{25}} \\
&= 4\sqrt{5} - 17\sqrt{5} + 11\sqrt{5} + \tfrac{13}{5}\sqrt{5} \\
&= 17\tfrac{3}{5}\sqrt{5} - 17\sqrt{5} = \tfrac{3}{5}\sqrt{5}.
\end{aligned}$$

(2) Multiply $3\sqrt{5} + 4\sqrt{2} - 5\sqrt{3}$ by $3\sqrt{5} + 4\sqrt{2} + 5\sqrt{3}$.

$$\begin{array}{l}
3\sqrt{5} + 4\sqrt{2} - 5\sqrt{3} \\
3\sqrt{5} + 4\sqrt{2} + 5\sqrt{3} \\
\hline
45 + 12\sqrt{10} - 15\sqrt{15} \\
\qquad 12\sqrt{10} + 32 - 20\sqrt{6} \\
\qquad\qquad\qquad 15\sqrt{15} + 20\sqrt{6} - 75 \\
\hline
\qquad\qquad 2 + 24\sqrt{10}.
\end{array}$$

Or, by formula,

$$\begin{aligned}
&(3\sqrt{5} + 4\sqrt{2} - 5\sqrt{3})(3\sqrt{5} + 4\sqrt{2} + 5\sqrt{3}) \\
&= (\overline{3\sqrt{5} + 4\sqrt{2}} - 5\sqrt{3})(\overline{3\sqrt{5} + 4\sqrt{2}} + 5\sqrt{3}) \\
&= (3\sqrt{5} + 4\sqrt{2})^2 - (5\sqrt{3})^2 \\
&= 45 + 32 + 24\sqrt{10} - 75 = 2 + 24\sqrt{10}.
\end{aligned}$$

(3) Simplify $\dfrac{p-q}{\sqrt{p}-\sqrt{q}} - \dfrac{p+q}{\sqrt{p}+\sqrt{q}}$.

$$\begin{aligned}
&\frac{p-q}{\sqrt{p}-\sqrt{q}} - \frac{p+q}{\sqrt{p}+\sqrt{q}} \\
&= \frac{(p-q)(\sqrt{p}+\sqrt{q}) - (p+q)(\sqrt{p}-\sqrt{q})}{p-q} \\
&= \frac{p(\sqrt{p}+\sqrt{q}-\sqrt{p}+\sqrt{q}) - q(\sqrt{p}+\sqrt{q}+\sqrt{p}-\sqrt{q})}{p-q} \\
&= \frac{2p\sqrt{q} - 2q\sqrt{p}}{p-q}.
\end{aligned}$$

Exercise XLIII.

Simplify

(1) $\sqrt{147} - \sqrt{243} - \sqrt{507} + 2\sqrt{108}$.

(2) $\sqrt[3]{56} + \sqrt[3]{1512} - \sqrt[3]{5103} - \sqrt[3]{875}$.

(3) $\sqrt{9\frac{1}{7}} - \sqrt{\frac{144}{567}} + \sqrt{\frac{169}{343}} - \sqrt[4]{\frac{256}{784}}$.

(4) $\sqrt{a^3 + a^2b + a^2c} + \sqrt{ab^2 + b^3 + b^2c} + \sqrt{ac^2 + bc^2 + c^3}$.

(5) $\sqrt{(a^2p + 18ap + 81p)} + \sqrt{(9py^2 - 6apy + a^2p)} - \sqrt{(ap + px - 2a^{\frac{1}{2}}x^{\frac{1}{2}}p)}$.

(6) $\sqrt[3]{2r^3x^p} + \sqrt[3]{2x^{4p+9}} - \sqrt[3]{16x^{p-3}y^6} + \sqrt[3]{54x^{p+6}y^3}$.

(7) $\{\sqrt[18]{(6p^2q^3 - 126p^2q^2 + 882p^2q - 2058p^2)}\}^6 \times \sqrt{\left\{\dfrac{q^2}{q^2 - 14q + 49}\right\}}$.

Write the following surds in the order of magnitude, and give the results by which they are compared:

(8) $5\sqrt[3]{7}$, $4\sqrt{10}$. (9) $6\sqrt[4]{18}$, $3\sqrt{18}$.

(10) $7\sqrt[3]{90}$, $13\sqrt[3]{9}$, $9\sqrt[3]{65}$, $17\sqrt[3]{11}$.

(11) $8\sqrt[3]{\frac{1}{5}}$, $4\sqrt[6]{\frac{1}{27}}$, $3\sqrt{\frac{2}{3}}$. (12) $\sqrt[4]{\frac{1}{3}}$, $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{1}{5}}$, $\sqrt[6]{\frac{1}{10}}$.

Multiply

(13) $4\sqrt{3} + 5\sqrt{2} - 6\sqrt{5}$ by $4\sqrt{3} + 6\sqrt{5} - 5\sqrt{2}$.

(14) $\sqrt{3} + 7\sqrt{8} - \sqrt{6}$ by $\sqrt{6} - \sqrt{3}$.

(15) $\sqrt{11} - \sqrt{13} + \sqrt{7}$ by $-\sqrt{7} + \sqrt{11} + \sqrt{13}$.

(16) $\sqrt{6} + \sqrt{11} - \sqrt{13}$ by $-\sqrt{11} - \sqrt{6} - \sqrt{13}$.

(17) $\sqrt{a} - \sqrt[4]{ab} + \sqrt{b}$ by $\sqrt{a} + \sqrt[4]{ab} + \sqrt{b}$.

(18) $\sqrt{a} + \sqrt[3]{a^2} - \sqrt[4]{a^3}$ by $\sqrt{a} - \sqrt[3]{a^2} + \sqrt[4]{a^3}$.

(19) Find the continued product of $\sqrt{3} - 2\sqrt{2}$, $4 - \sqrt{7}$, $-\sqrt{3} + \sqrt{6}$, $\sqrt{6} + \sqrt{3}$, $\sqrt{7} + 4$, $2\sqrt{2} + \sqrt{3}$.

Divide

(20) $\sqrt{35} + 12\sqrt{12} - 5\sqrt{15} + 90\sqrt{2} - 6\sqrt{42} - 2\sqrt{10} + 3\sqrt{30} - 108$ by $\sqrt{7} - \sqrt{8} + 3\sqrt{6} - 5\sqrt{3}$.

(21) $-2\sqrt{42} - 7\sqrt{30} + \sqrt{35} + 46$ by $\sqrt{7} - 3\sqrt{6} + 2\sqrt{5}$.

Extract the *square* roots of the following binomial surds:

(22) $20 - 2\sqrt{91}$. (23) $38 + 6\sqrt{30}$.

(24) $1\frac{1}{12} - \frac{1}{6}\sqrt{30}$. (25) $8 - 2(15)^{\frac{1}{2}}$.

(26) $2\{a - \sqrt{(a^2 - x^2)}\}$.

(27) $\frac{2}{3}\{m^2 + n^2 - \sqrt{(m^4 + m^2n^2 + n^4)}\}$.

Also obtain the *fourth* roots of

(28) $49 - 20\sqrt{6}$. (29) $184 - 40(21)^{\frac{1}{2}}$.

Express the following fractions with rational denominators, and in their simplest forms:

(30) $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$. (31) $\dfrac{\sqrt{19} + \sqrt{7}}{\sqrt{19} - \sqrt{7}}$.

(32) $\dfrac{1 - \sqrt{2}}{1 - \sqrt{2} + \sqrt{3}}$. (33) $\dfrac{\sqrt{3} - 1}{2\sqrt{2} + \sqrt{3} + 1}$.

(34) $\dfrac{\sqrt{3} + 1}{2\sqrt{2} - 1 + \sqrt{3}}$. (35) $\dfrac{3\sqrt{3} - \sqrt{5}}{3\sqrt{3} - \sqrt{2} + \sqrt{5}}$.

(36) $\dfrac{(\sqrt{x} - 1) + (\sqrt{x} + 1)}{(\sqrt{x} + 1) - (\sqrt{x} - 1)}$.

(37) $\dfrac{\sqrt{(x - 1)} + \sqrt{(x + 1)}}{\sqrt{(x + 1)} - \sqrt{(x - 1)}}$.

(38) $\dfrac{\sqrt{(1 + p)}}{\sqrt{(1 - p^2)}}\left\{\dfrac{\sqrt{(1 - p)} + \dfrac{1}{\sqrt{(1 + p)}}}{1 + \dfrac{1}{\sqrt{(1 - p^2)}}}\right\}$.

(39) $\dfrac{\sqrt{(p^2 + q^2)} + \sqrt{(p^2 - q^2)}}{\sqrt{(p^2 + q^2)} - \sqrt{(p^2 - q^2)}} + \dfrac{\sqrt{(p^2 + q^2)} - \sqrt{(p^2 - q^2)}}{\sqrt{(p^2 + q^2)} + \sqrt{(p^2 - q^2)}}$.

Solve the following equations:

(40) $x - 3\sqrt{x} + 18 = (\sqrt{x} + 3)^2 - (5\sqrt{x} + 3)$.

(41) $2\frac{1}{2}x - \frac{1}{2}(\sqrt{x} - 2) - [2\sqrt{x} - (\frac{1}{4}\sqrt{x} - \frac{1}{16}\{16 - \frac{1}{2}(\sqrt{x} + 4)\})]$
$= \frac{3}{2}(\sqrt{x} + 2 + \frac{5}{3}x) - 4\sqrt{x}$.

(42) $\sqrt{\dfrac{1}{p} - \dfrac{1}{x}} = \left\{\sqrt{\dfrac{1}{p^2} - \dfrac{1}{x}\sqrt{\dfrac{4}{p^2} - \dfrac{7}{x^2}}}\right\}^{\frac{1}{2}}$.

(43) $\frac{1}{2}\{x - 10\frac{25}{26} + \sqrt{x} - (x - 9)\} - \dfrac{2 - 6\sqrt{x}}{13}$

$= \sqrt{x} - \frac{1}{39}\{5\sqrt{x} + 3x - \frac{4}{3}(1 - 8\sqrt{x} + \frac{9}{8}x)\}.$

(44) $\sqrt{y^2 + 2py} + \sqrt{y^2 - 2py} = \dfrac{py}{\sqrt{y^2 + 2py}}.$

QUADRATIC EQUATIONS.

89. A QUADRATIC EQUATION is one which, when reduced to its simplest form, contains only the *second power*, or only the *second* and *first powers*, of the unknown quantity; as $x^2=4$, $x^2-6x=16$. See also sect. 103.

Quadratic Equations are also called *Equations of the Second Degree*, or *Equations of Two Dimensions*.

A PURE QUADRATIC EQUATION is one which, when reduced to its simplest form, contains only the *second power* of the unknown quantity; as $x^2 = 4$.

AN AFFECTED QUADRATIC EQUATION is one which, when reduced to its simplest form, contains both the *second and first powers* of the unknown quantity; as $x^2 + 4x = 45$, $x^2 - 6x = 16$.

If two quantities are equal, their square roots are equal. This is the proposition on which the solution of a quadratic depends.

Thus, if $x^2 = 9$, then $\pm x = \pm 3$; if $x^2 + 6x + 9 = 25$, then $\pm (x + 3) = \pm 5$.

SOLUTION OF QUADRATIC EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

I. PURE QUADRATICS.

90. A Pure Quadratic is in its *simplest form* when it is of the form

$$x^2 = \pm a;$$

that is, when we have, on *the left-hand side*, only x^2 with 1 for its coefficient and *plus* for its sign; and, on *the right-hand side*, only some known quantity with either sign.

Thus, $x^2=16$, $x^2=-9$, are in their simplest form; $3x^2 = 108$, $\dfrac{x^2}{3}=12$, $9-x^2=5$, $2x^2-7=x^2+2$, $(x+1)(x-1)=35$, are not.

91. TO REDUCE A PURE QUADRATIC TO ITS SIMPLEST FORM.—(1) Multiply out, if it contain factors; (2) multiply each term by the L.C.D., if it contain fractions; (3) transpose terms containing unknown quantities to the left-hand side, and known quantities to the other; (4) collect like quantities; (5) change the sign of each term if that of x^2 be *minus;* (6) divide by the coefficient of x^2.

EXERCISE XLIV.

Tell, from inspection, the simplest form of—

1. $x^2-4=0$; $x^2-8=8$; $9-x^2=0$; $x^2=6\frac{1}{4}$; $2x^2=8$.
2. $3x^2=75$; $54-6x^2=0$; $2x^2-9=63$; $9-2x^2=7$.
3. $20-3x^2=8$; $3x^2=9$; $2x^2-9=x^2+7$; $3x^2+1=5-x^2$.
4. $\frac{x^2}{2}=18$; $\frac{1}{x^2}=1$; $\frac{2x^2}{3}=6$; $\frac{48}{x^2}=3$; $x.\frac{x}{4}=4$; $\frac{x}{3}-\frac{3}{x}=0$.

92. TO SOLVE A PURE QUADRATIC EQUATION.—(1) Reduce it to its simplest form; (2) extract the square root of each side; (3) affix the double sign ($\pm$) to the square root of the known quantity.

Examples.

1. Solve $5-x^2=1$.

Transposing, $-x^2=-4$,

Changing signs, $x^2=4$,

Extracting, $x=\pm 2$.

2. $\frac{1}{2}-\frac{17}{x(x-3)}=\frac{3}{2x}-\frac{3}{x-3}$.

Multiplying by $2x(x-3)$,

$x(x-3)-34=3(x-3)-3.2x$,

$\therefore x^2-3x-34=3x-9-6x$,

$\therefore x^2=25$,

$\therefore x=\pm 5$.

3. Solve $\frac{2x}{x-3}-\frac{x}{x-4}=\frac{5}{7}$.

Multiplying by $7(x-3)(x-4)$,

$14x(x-4)-7x(x-3)$

$=5(x-3)(x-4)$,

whence $2x^2=60$,

$\therefore$, dividing by 2, $x^2=30$,

$\therefore x=\pm\sqrt{30}$.

4. Solve $x(x+3)(x-3)=7x$.

Dividing by x, $(x+3)(x-3)=7$,

whence $x^2=16$, $\therefore x=\pm 4$,

and, since we divided by x, $x=0$.

Note 1. In (1) we changed signs, because x^2 must be *plus;* in (3) we divided by 2, because the coefficient of x^2 must be 1.

Note 2. In (4), if we take $x=0$, the equation holds good, since then both sides become 0. Therefore, when we divide both sides by x, we must write 0 as one of the values of x.

Note 3. In (1) we might write $\pm x = \pm 2$, but we should get no new result by doing so. For, if we write $-x = \pm 2$, then $x = \mp 2$, the same values as before.

Note 4. In (3) we may find $\sqrt{30}$ approximately by arithmetic.

Exercise XLV.

1. $x^2=49$; $x^2-9=16$; $16-x^2=0$; $x^2=2\frac{1}{4}$; $2x^2=8$.
2. $41-x^2=5$; $8-x^2=1\frac{3}{4}$; $3x^2=6$; $27-3x^2=0$.
3. $15-7x^2=8$; $4-3x^2=2\frac{2}{3}$; $x(x-3)=25-3x$.
4. $3x(x-2)=147-6x$; $x(4-2x)=4x-2$; $(x+7)(7-x)=24$.
5. $(x+9)(x-4)=5x$; $(x-2)(x+3)=6x$; $(x-1)^2=32-2x$.
6. $x(x-1)(x+1)=3x$; $(3-x)(4-x)=16-7x$.
7. $\dfrac{x^2}{2}=8$; $\dfrac{2x^2}{3}-24=0$; $\dfrac{32}{x^2}=2$; $\dfrac{x}{8}=\dfrac{8}{x}$; $x-\dfrac{9}{x}=0$.
8. Solve the equations in Exercise XLIV.
9. $\dfrac{24}{x}=\dfrac{3x}{2}$; $\dfrac{x}{5}-\dfrac{5}{x}=0$; $x^2-\dfrac{1}{9}=\dfrac{1}{3}$; $\dfrac{5}{8}-x^2=\dfrac{1}{16}$.
10. $\dfrac{4x}{3}-\dfrac{12}{x}=x$; $\dfrac{x}{2}-\dfrac{2}{x}=\dfrac{4x}{9}$; $\dfrac{x}{4}-\dfrac{4}{x}=\dfrac{x}{8}+\dfrac{28}{x}$.
11. $\dfrac{5}{2x^2}-\dfrac{4}{3x^2}=\dfrac{14}{3}$; $\dfrac{9}{2x}+\dfrac{2x}{3}=\dfrac{3x}{4}-\dfrac{7\frac{1}{2}}{x}$; $\dfrac{x^2-1}{8}=\dfrac{x^2+9}{9}$.
12. $\dfrac{1}{x^2-1}=\dfrac{3}{x^2+1}$; $\dfrac{3}{5x^2-2}=\dfrac{5}{6-x^2}$; $\dfrac{x}{3}+\dfrac{2}{x}+15=\dfrac{5}{3}\cdot\dfrac{3+9x}{x}$.
13. $\dfrac{28}{3x^2+2}=\dfrac{32}{5x^2-4}$; $\dfrac{x^2-4}{5}-1=\dfrac{x^2-1}{6}$; $\dfrac{4}{x+5}+\dfrac{x+5}{4}=\dfrac{5}{2}$.
14. $\frac{2}{7}(x^2-4)=3(x^2+6)-5$; $\frac{5}{16}(3x^2-2)-6=x^2-\frac{1}{3}(x^2-4)$.
15. $\dfrac{6}{3x^2-2}-\dfrac{1}{2-3x^2}=7$; $\dfrac{x}{2}+\dfrac{x}{3}+\dfrac{4}{x}=\dfrac{x}{4}+\dfrac{6}{x}+\dfrac{2}{6x}$.
16. $\dfrac{3x}{4}-\dfrac{5x}{12}-\dfrac{2}{3x}=\dfrac{9+x}{9}-\dfrac{2(3x-4)}{6x}$; $\dfrac{3}{x+2}-\dfrac{3}{x-2}=4$.
17. $\dfrac{3}{x-2}-\dfrac{3}{2+x}=\dfrac{1}{5}$; $\dfrac{3x}{x+16}+\dfrac{3x}{x+36}=3$.
18. $3\dfrac{2x^2+3}{7}-3\dfrac{2-\frac{1}{3}x^2}{5}=\dfrac{9x^2}{11}+29$; $\dfrac{285}{4x^2-5}=\dfrac{141}{2x^2-3}$.
19. $\dfrac{5x}{8}-\dfrac{3(x-2)}{2x}+\dfrac{2x-15\frac{1}{3}}{4x}=\dfrac{3x-11}{6}-\dfrac{3x-10}{12}+\dfrac{2}{3x}$.

20. $\frac{x+3}{x-3}+\frac{x-3}{x+3}=4\frac{1}{4}$; $2\frac{x+1}{x-3}+3\frac{x-3}{x+1}=7$; $\frac{x+1}{x-2}+\frac{2x+5}{x+2}=\frac{3x+2}{x-1}$.

21. $\frac{x-1}{x+2}+\frac{2x-5}{x-2}=\frac{3x-2}{x+1}$; $\frac{5}{2x}=\frac{3}{x-5}+\frac{2}{x+4}-\frac{3(2x+3)}{2(x+4)(x-5)}$.

II. Adfected Quadratics.

93. In solving an Adfected Quadratic Equation by the usual method, there are three steps—namely:

First Step: Reduction to Simplest Form.—An Adfected Quadratic Equation is in its *simplest form* when it is of the form

$$x^2 \pm ax = \pm b;$$

that is, when we have, on *the left-hand side*, only (1) x^2 with 1 for its coefficient and *plus* for its sign, and (2) x with any known quantity for its coefficient and with either sign; and, on *the right-hand side*, only some known quantity with either sign.

Thus, $x^2+6x=27$, $x^2-3x=-2$, $x^2+\frac{2x}{3}=11$, $x^2-\frac{3x}{4}=13$, are in their simplest form; $3x^2+2x=33$, $4x-x^2=3$, $x^2-7=5-x$, are not.

94. To Reduce an Adfected Quadratic Equation to its Simplest Form.—Proceed as for a Pure Quadratic (sect. 91).

Exercise XLVI.

Tell, from inspection, the simplest form of

1. $x^2-4x+4=0$; $2x-x^2=-15$; $-x^2+5x=-24$.
2. $8x-15=x^2$; $6x=16-x^2$; $x^2=16+6x$; $-x^2-8x=12$.
3. $2x^2+8x=24$; $3x^2-6x=9$; $2x^2=5x-2$; $6x^2-5x=4$.
4. $4x^2+1=4x$; $9x^2+3x=2$; $5x-6x^2=1$; $2=7x-6x^2$.
5. $x^2=\frac{x}{15}+\frac{2}{5}$; $\frac{3x}{4}=\frac{9}{8}-x^2$; $x^2+\frac{1}{2}=\frac{3x}{2}$; $\frac{1}{2}-x^2=\frac{x}{2}$.

95. *Second Step: Completing the Square.*—To solve an Adfected Quadratic we must be able to extract the square root of its left-hand side; and, if that side is not a complete algebraic square, we must make it so. Now, an expression of the form x^2+ax, or x^2-ax, such as we find on the left-hand side of an adfected quadratic in its simplest form, is not a complete algebraic square. If it were the square of a single term (as x, $2x$), it would itself be a single term (as x^2, $4x^2$); if the square of a

binomial (as $x+a$, $x-a$), it would (sect. 33. 1) contain three terms (as $x^2+2ax+a^2$, $x^2-2ax+a^2$).

But, since $$\left(x+\frac{a}{2}\right)^2 = x^2+ax+\frac{a^2}{4},$$

and $$\left(x-\frac{a}{2}\right)^2 = x^2-ax+\frac{a^2}{4},$$

we see that to make x^2+ax or x^2-ax a complete algebraic square, we have only to add to it $\frac{a^2}{4}$; that is $\left(\frac{a}{2}\right)^2$; that is, the square of the half of a; that is, the square of half the coefficient of x in x^2+ax or x^2-ax. Therefore,

96. To Make the Left-hand Side of an Adfected Quadratic (in its Simplest Form) a complete Algebraic Square.—Add to it the square of half the coefficient of x.

Note. When the coefficient of x is an *odd* number, we get its half by writing 2 under that number; when it is a *fraction*, by dividing the numerator, or multiplying the denominator, by 2 (sect. 55. 2).

Exercise XLVII.

Tell, from inspection, what must be added to the following expressions to make them complete algebraic squares:

1. x^2+6x; x^2+2x; x^2+12x; x^2+38x; x^2+56x; x^2+92x.
2. x^2-2x; x^2-8x; x^2-4x; x^2-52x; x^2-112x; x^2-178x.
3. x^2+3x; x^2+5x; x^2+11x; x^2+x; x^2-x; x^2-9x.
4. $x^2+\frac{2}{3}x$; $x^2+\frac{4}{5}x$; $x^2+\frac{8x}{9}$; $x^2-\frac{2}{5}x$; $x^2-\frac{6x}{7}$; $x^2-\frac{12x}{11}$.
5. $x^2+\frac{1}{3}x$; $x^2-\frac{3}{4}x$; $x^2+\frac{x}{6}$; $x^2+\frac{7x}{4}$; $x^2-\frac{x}{2}$; $x^2-\frac{12x}{30}$.

97. *Third Step: Extracting the Square Root.*

Since $x^2+ax+\frac{a^2}{4}$ and $x^2-ax+\frac{a^2}{4}$ are complete algebraic squares, whose square roots are $x+\frac{a}{2}$, $x-\frac{a}{2}$,

we see that, to write from inspection the square roots of these and similar squares, we have only to take *the square root of the first term* and the *square root of the third term* of the complete square, and to give the latter *the sign of the second term* of the square.

EXERCISE XLVIII.

Tell, from inspection, the square roots of

1. x^2+2x+1; x^2-6x+9; x^2-4x+4; $x^2-10x+5^2$.
2. $x^2-12x+6^2$; $x^2-58x+29^2$; $x^2+34x+289$; $x^2-24x+144$.
3. $x^2+x+\left(\frac{1}{2}\right)^2$; $x^2-3x+\frac{9}{4}$; $x^2-\frac{3x}{2}+\left(\frac{3}{4}\right)^2$; $x^2+\frac{5x}{7}+\frac{25}{196}$.
4. $x^2+\frac{2x}{3}+\frac{1}{9}$; $x^2-\frac{4x}{5}+\frac{4}{25}$; $x^2-\frac{18x}{11}+\frac{81}{121}$; $x^2+\frac{24x}{18}+\frac{16}{36}$.
5. $x^2-\frac{35x}{48}+\left(\frac{35}{96}\right)^2$; $x^2-\frac{5x}{4}+\frac{25}{64}$; $x^2-\frac{7x}{3}+\frac{49}{36}$; $x^2-\frac{9x}{5}+\frac{81}{100}$.

98. TO SOLVE AN AFFECTED QUADRATIC EQUATION.—RULE I. (1) Reduce it to its simplest form; (2) add to each side *the square of half the coefficient of x;* (3) extract the square root of each side; (4) affix the double sign ($\pm$) to the square root of the known quantity; (5) solve the two resulting simple equations. This method is called *Completing the Square.*

Examples.

1. Solve $x^2+6x=16$.

Completing the square,

$$x^2+6x+3^2=16+9=25;$$

Extracting square root,

$$x+3=\pm 5;$$

Solving these simple equations,

$$x=-3+5=2,$$

or $x=-3-5=-8$.

2. $x^2-8x=-16$.

$\therefore\ x^2-8x+4^2=16-16=0,$

$\therefore\ x-4=\pm 0,$

$\therefore\ x=4.$

3. $4x-x^2=7$.

$\therefore\ x^2-4x=-7,$

$\therefore\ x^2-4x+2^2=4-7=-3,$

$\therefore\ x=2\pm\sqrt{-3}.$

4. $x^2-5x=6$.

$\therefore\ x^2-5x+\left(\frac{5}{2}\right)^2=\frac{25}{4}+\frac{24}{4}=\frac{49}{4}.$

$\therefore\ x-\frac{5}{2}=\pm\frac{7}{2},$

$\therefore\ x=\frac{5+7}{2}$ or $\frac{5-7}{2}=6$ or -1.

5. $x^2-x-20=0$.

$\therefore\ x^2-x=20,$

$\therefore\ x^2-x+\left(\frac{1}{2}\right)^2=\frac{1+80}{2^2}=\frac{81}{2^2},$

$\therefore\ x=\frac{1\pm 9}{2}=5$ or -4.

6. $x^2 + \frac{2x}{3} = \frac{7}{12}$.

$\therefore\ x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{1}{9} + \frac{7}{12} = \frac{25}{36}$,

$\therefore\ x = -\frac{1}{3} \pm \frac{5}{6}$,

$= \frac{-2 \pm 5}{6} = \frac{1}{2}$ or $-\frac{7}{6}$.

7. $x^2 - \frac{x}{5} = \frac{3}{20}$.

$\therefore\ x^2 - \frac{1}{5}x + \left(\frac{1}{10}\right)^2 = \frac{1+15}{10^2} = \frac{16}{10^2}$,

$\therefore\ x = \frac{1 \pm 4}{10} = \frac{1}{2}, -\frac{3}{10}$.

8. $2x^2 - 3x = 35$.

$\therefore\ x^2 - \frac{3x}{2} = \frac{35}{2}$,

$\therefore\ x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{9+280}{4^2}$

$= \frac{289}{4^2}$,

$\therefore\ x = \frac{3 \pm 17}{4} = 5, -3\frac{1}{2}$.

9. $x - \frac{x^2}{3} = -6$.

$\therefore\ x^2 - 3x = 18$.

$\therefore\ x^2 - 3x + \left(\frac{3}{2}\right)^2 = \frac{9+72}{2^2} = \frac{81}{2^2}$,

$\therefore\ x = \frac{3 \pm 9}{2} = 6, -3$.

Note 1. To reduce the equation to its simplest form (sect. 93, 94), we change signs in (3), transpose in (5), divide by 2 in (8), and multiply by 3 in (9).

Note 2. In (4), (5), and (9), where the coefficient of x is an odd number, we get its half by writing 2 under it; in (6), where it is a fraction with an even numerator, we divide the numerator by 2; in (7) and (8), where it is a fraction with an odd numerator, we multiply the denominator by 2 (sect. 96, note).

Note 3. It is better merely to indicate the squaring of the added term, on the left-hand side; and of its denominator, on the right-hand side. And the pupil should accustom himself to taking the last two steps of the work (extracting, and solving the two simple equations) at once, as in (3), (5), and following examples.

Note 4. In (2), the two values of x, ($x=4+0$, and $x=4-0$), are equal, both being 4. In (3), since (sect. 78. 2) we cannot find $\sqrt{-3}$, we cannot solve the equation in numbers; that is, there is no number such that $4x - x^2 = 7$. But, had we found $x = 2 \pm \sqrt{3}$, we might have solved it approximately by arithmetic.

Note 5. In (1) we might write $\pm(x + 3) = \pm 5$, but this would give us no new result. For, if we write $-(x + 3) = \pm 5$, then $+(x + 3) = \mp 5$, as before (compare sect. 92, note 3).

10. $\frac{4}{3x} - 3\frac{x-4}{4} - x = \frac{2x-\frac{1}{3}}{2x} - \frac{3}{4}$.

Multiplying by $12x$, the L.C.D.

$$4.4 - 3.3x(x-4) - 12x^2$$
$$= 6(2x - \tfrac{1}{3}) - 3.3x;$$
$$\therefore\ 16 - 9x^2 + 36x - 12x^2$$
$$= 12x - 2 - 9x,$$
$$\therefore\quad 21x^2 - 33x = 18,$$
$$\therefore\quad x^2 - \frac{11x}{7} = \frac{6}{7},$$
$$\therefore\quad x^2 - \frac{11x}{7} + \left(\frac{11}{14}\right)^2 = \frac{289}{14^2},$$
$$\therefore\quad x = \frac{11 \pm 17}{14} = 2, -\frac{3}{7}.$$

11. $\frac{15}{2(x^2-4)} - \frac{4}{3(x+2)} = \frac{1}{6}$.

Multiplying by $6(x^2 - 4)$,

$$3.15 - 4.2(x-2) = x^2 - 4;$$
$$\therefore\ x^2 + 8x = 65;\ \therefore\ x = 5, -13.$$

12. $\frac{x+1}{x+3} + \frac{2x-5}{x-1} = \frac{3x-1}{x+2}$.

$$\therefore\quad (x+1)(x-1)(x+2)$$
$$+ (2x-5)(x+3)(x+2)$$
$$= (3x-1)(x+3)(x-1),$$
$$\therefore\quad x^3 + 2x^2 - x - 2$$
$$+ 2x^3 + 5x^2 - 13x - 30$$
$$= 3x^3 + 5x^2 - 11x + 3,$$
$$\therefore\ 2x^2 - 3x = 35, \text{ solved in (8).}$$

Exercise XLIX.

1. $x^2 + 6x = 40$; $x^2 + 8x = 65$; $x^2 + 2x = 35$; $x^2 + 10x = 0$.
2. $x^2 + 4x = 96$; $x^2 + 12x = -36$; $x^2 + 16x = -55$; $x^2 + 4x = 60$.
3. $x^2 - 6x = 7$; $x^2 - 2x = 24$; $14x - x^2 = 49$; $x^2 - 8x = 128$.
4. $x^2 - 2x = 15$; $20x - x^2 = 99$; $18x - x^2 = 81$; $x^2 - 4x = 21$.
5. $x^2 + 3x = 10$; $x^2 + 7x = -12$; $x^2 + 9x = -20$; $x^2 + x = 2$
6. $x^2 + x = 12$; $x^2 + 5x = 24$; $x^2 + 13x = 30$; $x^2 + x = \frac{3}{4}$.
7. $x^2 - 3x = -2$; $5x - x^2 = 6$; $x^2 - 9x = 22$; $x^2 - 11x = 42$.
8. $x^2 - 15x = 0$; $x^2 - x = 2$; $x^2 - x = \frac{5}{16}$; $x - x^2 = \frac{2}{9}$.
9. $x^2 + \frac{6}{5}x = \frac{16}{25}$; $x^2 + \frac{2}{3}x = -\frac{1}{9}$; $3x^2 + 4x = 4$; $x^2 + \frac{8}{3}x = -\frac{5}{3}$.
10. $x^2 + \frac{10x}{7} = \frac{24}{49}$; $x^2 + \frac{14x}{9} = \frac{40}{27}$; $x^2 + \frac{16x}{15} = 1$; $x^2 + \frac{8x}{21} = 2\frac{1}{7}$.
11. $x^2 - \frac{2x}{9} = \frac{8}{81}$; $\frac{6x}{5} - x^2 = \frac{1}{5}$; $3x^2 - 4x = 15$; $\frac{16x}{15} - x^2 = \frac{4}{15}$.
12. $x^2 - \frac{10x}{9} = \frac{8}{27}$; $x^2 - \frac{4x}{15} = \frac{1}{5}$; $\frac{2x}{3} - x^2 = \frac{8}{81}$; $2x - 3x^2 = \frac{1}{4}$.
13. $x^2 + \frac{3x}{2} = 1$; $x^2 + \frac{x}{4} = \frac{3}{8}$; $x^2 + \frac{x}{3} = \frac{1}{12}$; $6x^2 + x = 2$.
14. $x^2 + \frac{7x}{4} = \frac{15}{8}$; $x^2 + \frac{11x}{3} = -3\frac{1}{3}$; $x^2 + \frac{5x}{12} = \frac{1}{4}$; $x^2 + \frac{3x}{2} = \frac{35}{18}$.

15. $x^2-\frac{3x}{2}=-\frac{9}{16}$; $2x^2-x=\frac{3}{8}$; $x^2-\frac{x}{3}=\frac{2}{9}$; $x^2-\frac{5x}{6}=\frac{7}{18}$.

16. $x^2-\frac{13x}{5}=\frac{14}{25}$; $\frac{11x}{8}-x^2=\frac{5}{32}$; $x^2-\frac{17x}{18}=\frac{19}{162}$; $\frac{x}{2}-x^2=\frac{1}{18}$.

17. Solve the Equations in Exercise XLVI.

18. $3x^2-2x=21$; $4x^2-x=33$; $6x^2-x=15$; $17x-2x^2=35$.

19. $(x+2)(6-x)=15$; $(3x+2)^2+(3x-8)^2=122$.

20. $(x-6)(x+4)=6x+9$; $(x+3)(x-3)=2x(x-5)$.

21. $x(7-x)+2x(x-2)=10(x-1)$; $5x^2-14=3x$.

22. $(x+7)^2-(x-3)^2=2(x+2)^2$; $3x^2-26x=169$.

23. $2(x+5)(x-5)=3(x-3)^2$; $3(2x-1)(x+2)=4(x+1)^2$.

24. $(4x-5)(x+4)-(3x-2)(2x-3)=14$.

25. $(x+3)(x-5)-(x-3)(7-x)-(x-6)(x+13)=0$.

26. $x+\frac{8}{x}=6$; $\frac{72-x}{x}=\frac{x}{6}$; $\frac{x}{x+36}=\frac{13}{2x+3}$; $\frac{x}{x-3}=\frac{51}{3x+7}$.

27. $\frac{x-1}{x+9}=\frac{9}{2(x-2)}$; $\frac{x}{2}+\frac{8}{x}=4$; $x-\frac{5}{x}=\frac{4}{3}$; $x+\frac{6}{x}=\frac{11}{2}$.

28. $x-\frac{15}{2x}=\frac{15}{2x}-2$; $4x-\frac{8}{3x}=\frac{20}{3}$; $x-\frac{x-1}{x-4}=5$.

29. $3x-2\frac{x+4}{x-2}=9$; $\frac{x}{3}+\frac{3}{x}=\frac{x}{4}+\frac{4}{x}+\frac{1}{3}$; $\frac{2x}{3}-\frac{3}{2x}=\frac{6}{x}-\frac{1}{2}$.

30. $\frac{2x}{3}-\frac{4x-3}{2x}=\frac{1}{2}$; $\frac{3x}{4}-\frac{9}{2x}=\frac{2x}{3}-\frac{6}{x}+1\frac{1}{8}$; $\frac{12}{x}-\frac{1}{2}=\frac{12}{x+2}$.

31. $\frac{20}{x}+4=\frac{20}{x-4}$; $\frac{3}{x}+\frac{5}{8-x}=2$; $\frac{3}{x+2}+\frac{1}{2}=\frac{2}{x-2}$.

32. $\frac{1}{x-4}-\frac{2}{x+4}=\frac{3}{10}$; $\frac{9}{x^2+6x+9}=\frac{4}{x^2-6x+9}$.

33. $\left(\frac{2}{x+2}\right)^2=\left(\frac{1}{x-2}\right)^2$; $\frac{16}{(x+4)^2}-\frac{9}{(x-4)^2}=0$.

34. $\frac{3x}{x+4}+\frac{x+4}{3x}=2$; $\frac{x}{x-1}-\frac{x}{x+3}=1$; $\frac{x+2}{3}-\frac{3}{x+2}=\frac{8}{3}$.

35. $\frac{x-2}{x-4}+\frac{x+1}{x-1}=3$; $\frac{2x+3}{3x}+\frac{6x}{3x-4}=3$; $\frac{x-2}{x-3}+\frac{2x-3}{x+9}=2$.

36. $\dfrac{3x-5}{x-7}-\dfrac{2x+9}{x+3}=4$; $\dfrac{x+3}{2x}-\dfrac{2x}{x+3}=\dfrac{3}{2}$; $\dfrac{3x}{x+2}-\dfrac{x+2}{3x}=\dfrac{3}{2}$.

37. $\dfrac{1}{x}+\dfrac{1}{x-4}=\dfrac{1}{2\frac{2}{3}}$; $\dfrac{5}{x}+\dfrac{1}{24}=\dfrac{5}{x-\frac{1}{5}}$; $\dfrac{3(x-1)}{4}-\dfrac{5(8-x)}{x-2}=x-7$.

38. $\dfrac{3x}{x+2}+\dfrac{x+2}{3x}=\dfrac{5}{2}$; $\dfrac{x}{x^2-4}=\dfrac{3(x-8)}{8(2-x)}$; $\dfrac{5}{x+3}+\dfrac{6}{x^2-9}=1$.

39. $\dfrac{5}{x^2-1}-\dfrac{1}{2(x-1)}=\dfrac{1}{6}$; $\dfrac{15}{2(x^2-4)}-\dfrac{4}{3(x+2)}=\dfrac{1}{6}$.

40. $\dfrac{3x+5}{7}-\dfrac{12-x}{x+3}=\dfrac{3x-2}{14}$; $\dfrac{x-4}{x+5}-\dfrac{x-9}{x+3}=\dfrac{1}{4}$.

41. $\dfrac{x-3}{x+3}-\dfrac{x-4}{x+6}=\dfrac{1}{6}$; $\dfrac{18}{x}-\dfrac{3}{5}=\dfrac{18}{x+1\frac{1}{2}}$; $\dfrac{4x-7}{2x}-\dfrac{3x+8}{3x-2}=\dfrac{1}{2}$.

42. $x\cdot\dfrac{4x}{3}=\left(\dfrac{5x}{3}-3\right)(x-2)$; $\dfrac{x-1}{x-3}+\dfrac{x-3}{x-1}=\dfrac{13}{6}$.

43. $\dfrac{x+2}{x+4}-\dfrac{x-4}{x-2}=\dfrac{3}{10}$; $\dfrac{x+3}{x+1}-\dfrac{x+1}{x+3}=\dfrac{5}{6}$; $\dfrac{x-8}{x-6}+\dfrac{x-8}{x+2}=\dfrac{4}{3}$.

44. $\dfrac{x+1}{x-7}-\dfrac{x+2}{x-4}=\dfrac{2}{3}$; $\dfrac{5}{x-3}-\dfrac{7}{x+2}-\dfrac{3x-2}{(x-3)(x+2)}=\dfrac{3}{4x}$.

45. $\dfrac{2(x+1)}{x-3}+\dfrac{5(x-3)}{x+1}=11$; $\dfrac{2x-5}{x+1}-\dfrac{x-8}{x-7}=\dfrac{2}{3}$.

46. $\dfrac{3x-7}{x+1}+\dfrac{x-1}{x-2}=\dfrac{8}{3}$; $\dfrac{3x+1}{2x-1}-\dfrac{2x+1}{3x-1}=\dfrac{4}{3}$; $\dfrac{2x+\frac{1}{2}}{3x-\frac{1}{4}}-\dfrac{x+\frac{1}{4}}{3x+\frac{1}{4}}=\dfrac{3}{2}$.

47. $\dfrac{x+1}{x-3}+\dfrac{x+3}{x-1}=\dfrac{2x+5}{x-2}$; $5x+\dfrac{7x+9}{4x+3}=9+\dfrac{10x^2-18}{2x+3}$.

48. $\dfrac{x+3}{x+1}+\dfrac{2x-1}{x+2}=\dfrac{3x-4}{x-1}$; $3x+\dfrac{7x-2}{3x+2}=6\left(1+\dfrac{x^2-1}{2x+3}\right)$.

49. $\dfrac{2x-5}{x-4}-\dfrac{x+2}{x-1}=\dfrac{x+5}{x+1}$; $x+1-3\cdot\dfrac{x-1}{2x-3}=3\dfrac{x^2-11}{3x-2}$.

50. $\dfrac{5x-1}{x+4}-\dfrac{3x-4}{x+2}=\dfrac{2x+1}{x+3}$; $x+\dfrac{x}{2}\cdot\dfrac{x+7}{2x-1}=\dfrac{5}{4}\cdot\dfrac{3(x^2+16)}{3x+1}$.

99. Rule II.—By Factoring. (1) Reduce the equation to its simplest form; (2) collect all its terms on one side; (3)

factorise the expression thus obtained; (4) write each factor equal to zero; (5) solve the resulting simple equations.

Examples.

1. $x^2 + 6x = 16$ (sect. 98. 1).
$x^2 + 6x - 16 = 0$.
$\therefore \ (x + 8)(x - 2) = 0$,
$\therefore$ either $x + 8 = 0, \ \therefore x = -8$;
or $x - 2 = 0, \ \therefore x = 2$.

2. $x^2 - 5x = 6$ (sect. 98. 4).
$x^2 - 5x - 6 = 0$,
$\therefore (x - 6)(x + 1) = 0$,
$\therefore$ either $x - 6 = 0, \therefore x = 6$;
or $x + 1 = 0, \therefore x = -1$.

3. $x^3 + 3x^2 - 4x = 0$,
$\therefore \ x(x^2 + 3x - 4) = 0$,
$\therefore \ x(x + 4)(x - 1) = 0$,
$\therefore$ either $x = 0$;
or $x + 4 = 0, \therefore x = -4$;
or $x - 1 = 0, \therefore x = 1$.

4. $3x^2 + 12x + 9 = 0$,
$\therefore \ 3(x^2 + 4x + 3) = 0$,
$\therefore \ 3(x + 3)(x + 1) = 0$,
$\therefore$ since 3 cannot $= 0$,
either $x + 3 = 0, \therefore x = -3$;
or $x + 1 = 0, \therefore x = -1$.

5. $2x^2 + 5x = -2$,
$\therefore \ 2x^2 + 5x + 2 = 0$,
$\therefore (2x + 1)(x + 2) = 0$,
$\therefore$ either $2x + 1 = 0$,
$\therefore 2x = -1, \therefore x = -\frac{1}{2}$;
or $x + 2 = 0, \therefore x = -2$.

Or, $2x^2 + 5x + 2 = 0$,
$\therefore$, dividing by 2,
$x^2 + \frac{5}{2} x + 1 = 0$,
$\therefore \ (x + 2)(x + \frac{1}{2}) = 0$,
$\therefore$ either $x + 2 = 0, \therefore x = -2$;
or $x + \frac{1}{2} = 0, \therefore x = -\frac{1}{2}$.

Note 1. This method depends on the fact that, if a product be equal to 0, one or more of its factors must be equal to 0.

It is a very easy method when the coefficient of x^2 is $+ 1$, and the other terms contain no fractions (as in examples 1 and 2). In other cases (as in example 5), the first method, or the third (sect. 100), should be preferred.

Note 2. In (4) only two of the factors can be equal to 0; in (3) all of them. Thus in (3) x has three values, but the equation is really one of the third degree (see sect. 103 and 92, *Note* 2).

Exercise L.

1. $x^2 - 5x + 6 = 0$; $x^2 + x - 12 = 0$; $x^3 - x^2 - 2x = 0$.
2. $2x^2 - 6x - 36 = 0$; $x^4 - 12x^3 + 32x^2 = 0$; $3x^3 - 6x^2 - 72x = 0$.
3. Solve by this method Exercise XLIX. 1—8.

100. Rule III.—By Formula. (1) Clear the equation of all fractions, and (2) arrange its terms in the following

$$\textit{Form.}\quad ax^2+bx+c=0,$$

$$\therefore x^2+\frac{bx}{a}=-\frac{c}{a},$$

$$\therefore x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},$$

$$\therefore x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a},$$

$$\textit{Formula.}\quad \therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Note 1. In the *Formula*, of course b^2 is always positive, and the radical quantity has always the double sign. But the signs of b and c differ in the *Formula* from their signs in the *Form*; so that, if either be negative in the *Form*, it will be positive in the *Formula*.

Note 2. If the expression be not cleared of fractions to begin with, fractions will appear in the *Formula*; and this should be avoided.

Note 3. If, after the equation is reduced, $a=1$,

the *Form* becomes $x^2+bx+c=0$;

and the *Formula*, $x=\dfrac{-b\pm\sqrt{b^2-4c}}{2}$.

Examples.

1. $x^2-x-20=0$ (sect. 98. 5).

Here $a=1$; $b=-1$,

$\therefore -b=1$; and

$-4ac=-4.1.-20=80$;

$$\therefore x=\frac{1\pm\sqrt{1^2+80}}{2}$$

$$=\frac{1\pm 9}{2}=5,\text{ or }-4.$$

2. $x^2+\dfrac{2x}{3}=\dfrac{7}{12}$ (sect. 98. 6),

$\therefore 12x^2+8x-7=0.$

Here $a=12$; $b=8$,

$\therefore -b=-8$; and

$-4ac=-4.12.-7=336$;

$$\therefore x=\frac{-8\pm\sqrt{8^2+336}}{2.12}=\frac{1}{2},-\frac{7}{6}.$$

Exercise LI.

1. Write the *Formulæ* (1) for $ax^2+bx-c=0$, $ax^2-bx+c=0$, $ax^2-bx-c=0$; (2) for $x^2+bx-c=0$, $x^2-bx+c=0$, $x^2-bx-c=0$.

2. $2x^2+3x=65$; $3x^2-50x=17$; $7x^2+81=72x$; $74-13x^2=11x$.

3. Solve in this way Exercise L. 2, 4, 6, 8, 18, 9, 13—15.

101. Since every Quadratic can be reduced to the form

$$x^2 + bx + c = 0,$$

and we have seen (sect. 100, *Note* 3) that then either

$$(1) \quad x = \frac{-b + \sqrt{b^2 - 4c}}{2} = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2},$$

$$\text{or } (2) \quad x = \frac{-b - \sqrt{b^2 - 4c}}{2} = -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2},$$

we see also that

1. A Quadratic equation has two roots (which may be equal), but only two; namely, the two values found above for x.

If $b = 0$, then $bx = 0$, and the equation is a pure quadratic. Thus the statement is true for all quadratics.

Similarly we might shew that an equation of the *third* degree has *three* roots; one of the *fourth* degree, *four* roots; and so on.

2. When a Quadratic has been reduced to the above form,
The *Sum of the Roots* = the coefficient of x with sign changed;
Their *Product* = the third term, with same sign.

For their *Sum* $= -\frac{b}{2} - \frac{b}{2} = -\frac{2b}{2} = -b$,

and their *Product* $= \frac{b^2}{4} - \frac{b\sqrt{b^2 - 4c}}{4} + \frac{b\sqrt{b^2 - 4c}}{4} - \frac{b^2 - 4c}{4}$

$$= \frac{b^2}{4} - \frac{b^2 - 4c}{4} = \frac{4c}{4} = c.$$

Thus we get a very useful *Test of Accuracy* for our work.

Examples.

1. $x^2 - x - 20 = 0$; $\therefore x = 5$ or -4 (sect. 98. 5).
Sum of Roots $= 5 - 4 = 1$, which is the coefficient of x with its sign changed;
Product of Roots $= -4.5 = -20$, which is the third term.

2. $2x^2 - 3x = 35$; $\therefore x = 5, -3\frac{1}{2}$ (sect. 98. 8).
$x^2 - \frac{3x}{2} - \frac{35}{2} = 0$, is the equation when reduced;
Sum of Roots $= 5 - 3\frac{1}{2} = 1\frac{1}{2} = \frac{3}{2}$;
Product of Roots $= -\frac{7}{2} \cdot \frac{5}{1} = -\frac{35}{2}$.

102. Equations like those in Exercise L. are easily solved if worked as in the following *Examples.* They come out Pure Quadratics, and may be taken immediately after Exercise XLVI.

1. $\frac{3}{x+3}-\frac{2}{x+9}=\frac{2}{x-9}-\frac{3}{x-3}$,

$\therefore \frac{3}{x+3}+\frac{3}{x-3}=\frac{2}{x+9}+\frac{2}{x-9}$,

$\therefore \frac{3x-9+3x+9}{x^2-9} = \frac{2x-18+2x+18}{x^2-81}$,

$\therefore \frac{6x}{x^2-9}=\frac{4x}{x^2-81}$,

$\therefore \frac{3}{x^2-9}=\frac{2}{x^2-81}$,

$\therefore x^2=225$, $\therefore x=\pm 15, 0$.

2. $\frac{x+1}{x-1}+\frac{9-x}{9+x}=\frac{9+x}{9-x}+\frac{x-1}{x+1}$,

$\therefore \frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{9+x}{9-x}-\frac{9-x}{9+x}$,

$\therefore \frac{x^2+2x+1-x^2+2x-1}{x^2-1} = \frac{81+18x+x^2-81+18x-x^2}{81-x^2}$,

$\therefore \frac{4x}{x^2-1}=\frac{36x}{81-x^2}$,

$\therefore \frac{1}{x^2-1}=\frac{9}{81-x^2}$,

$\therefore 10x^2=90$, $\therefore x=\pm 3, 0$.

Note.—In both cases we transpose in order to get like denominators on each side. In both cases, too, x was a factor of the equation, and we divided by x; therefore, $x=0$ is a solution of each (see sect. 92, *Note* 2).

Exercise LII.

1. $\frac{3}{x+1}+\frac{3}{x-1}=\frac{2}{x+3}+\frac{2}{x-3}$; $\frac{5}{x+5}+\frac{5}{x-5}-\frac{4}{x+10}-\frac{4}{x-10}=0$.

2. $\frac{1}{x+2}+\frac{1}{x-2}+\frac{1}{x+14}+\frac{1}{x-14}=0$; $\frac{2}{x+1}+\frac{2}{x-1}+\frac{3}{x+11}+\frac{3}{x-11}=0$; $\frac{1}{x+7}+\frac{1}{x-7}+\frac{1}{x+17}=\frac{1}{17-x}$.

3. $\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{4+x}{4-x}-\frac{4-x}{4+x}$; $\frac{x+4}{x-4}-\frac{x-4}{x+4}+\frac{16-x}{16+x}=\frac{16+x}{16-x}$.

4. $\frac{x+9}{x-9}-\frac{16+x}{16-x}=\frac{x-9}{x+9}-\frac{16-x}{16+x}$; $\frac{x+25}{x-25}+\frac{36-x}{36+x}=\frac{36+x}{36-x}+\frac{x-25}{x+25}$.

Equations Solvable as Quadratics.

103. Any equation can be solved as a Quadratic if (1) it contain only two powers of the unknown quantity, and (2) one of these powers be the square of the other.

Thus, since x^4 is the square of x^2, x^6 the square of x^3, x the square of $\sqrt{x}$, and x^2+8x+1 the square of $\sqrt{x^2+8x+1}$, the following examples can be solved as Quadratics by the methods already given:

Examples.

1. $x^4-6x^2=-8.$

$\therefore\ x^4-6x^2+3^2=1,$

$\therefore\ x^2=3\pm1=4$ or $2,$

$\therefore\ x=\pm2$ or $\pm\sqrt{2}.$

2. $x^6+26x^3=27.$

$\therefore\ x^6+26x^3+13^2=196,$

$\therefore\ x^3=-13\pm14=1$ or $-27,$

$\therefore\ x=1$ or $-3.$

3. $x+\sqrt{x}=12.$

$\therefore\ x+\sqrt{x}+\left(\frac{1}{2}\right)^2=\frac{49}{2^2},$

$\therefore\ \sqrt{x}=-\frac{1}{2}\pm\frac{7}{2}=+3,\ -4,$

$\therefore\ x=9,\ 16.$

4. $x-\sqrt{x}=12.$

$\therefore\ x-\sqrt{x}+\left(\frac{1}{2}\right)^2=\frac{49}{2^2},$

$\therefore\ \sqrt{x}=\frac{1}{2}\pm\frac{7}{2}=-3,\ +4,$

$\therefore\ x=9,\ 16.$

5. $x^2+8x-3+\sqrt{x^2+8x+1}=52.$

$\therefore\ x^2+8x+1+\sqrt{x^2+8x+1}=56,$

$\therefore\ x^2+8x+1+\sqrt{x^2+8x+1}+\left(\frac{1}{2}\right)^2=\frac{225}{2^2},$

$\therefore\ \sqrt{x^2+8x+1}=\frac{-1\pm15}{2}=+7,\ -8,$

$\therefore\quad x^2+8x+1=49,\ 64;$

Let (1) $x^2+8x+1=49,$

$\therefore\ x^2+8x=48,$

$\therefore\ x^2+8x+4^2=64,$

$\therefore\ x=-4\pm8,\ =+4,\ -12.$

Let (2) $x^2+8x+1=64,$

$\therefore\ x^2+8x=63,$

$\therefore\ x^2+8x+4^2=79,$

$\therefore\ x=-4\pm\sqrt{79}.$

Note 1. In (2), x seems to have only *two* roots. It really has *six*, as (sect. 101) it ought to have.

For, since $x^3=1$, $\therefore\ x^3-1=0$, $\therefore\ (x-1)(x^2+x+1)=0;$

$\therefore$ either $x-1=0$, or $x^2+x+1=0;$

and, by solving $x^2+x+1=0$, and $x^2-3x+9=0$ (got, in same way, from $x^3+27=0$), we get the other values of x.

Note 2. Example (4) differs from (3) only in the sign of $\sqrt{x}$. In both we get the same values for x, and thus, in solving the one, we solve the other also

This happens where we have to square a quantity before we can get the value of x. Thus, in (5), in solving $x^2+8x+1+$

$\sqrt{x^2+8x+1}=56$, we have also solved $x^2+8x+1-\sqrt{x^2+8x+1}=56$, as will be found on trial.

In verifying these equations we must be careful to take for the quantity under the root sign the value actually found for it. Thus, in (3), if $x=9$, then $\sqrt{x}=+3$ (not -3, which might also be got from $x=9$); in (4), if $x=9$, then $\sqrt{x}=-3$ (not $+3$); in (5), if $x^2+8x+1=64$, then $\sqrt{x^2+8x+1}=-8$ (not $+8$); and so on.

Note 3. In (5), we began by adding 4 to each side, that we might have, as the first term of the equation, x^2+8x+1, which is the square of $\sqrt{x^2+8x+1}$.

Exercise LIII.

1. $x^4-10x^2=-9$; $x^4-6x^2=27$; $x^4+576=52x^2$; $x^4+36=13x^2$.
2. $x^4-5x^2=-6$; $x^6-28x^3=-27$; $x^6-2x^3=48$.
3. $x^6+26x^3=27$; $x^6+56x^3=512$; $x^6-7x^3=8$; $x^6-19x^3=216$.
4. $x+2\sqrt{x}=24$; $x+\sqrt{x}=42$; $x-3\sqrt{x}=10$; $x-\sqrt{x}=0$.
5. $x+\sqrt{x+4}=8$; $x+4\sqrt{x+2}=30$; $x+5+\sqrt{x-2}=19$.
6. $3x-9-5\sqrt{x-5}=8$; $x^2+2\sqrt{x^2-3}=6$; $x^2+8-3\sqrt{x^2+12}=0$.
7. $x^2-6x+4\sqrt{x^2-6x+2}=19$; $x^2+x+2+2\sqrt{x^2+x+4}=22$.
8. $x^2-5x-3-\sqrt{x^2-5x+3}=0$; $(x+1)^2-\sqrt{x^2+2x-6}=13$.
9. $3x^2-12x-5+6\sqrt{x^2-4x-1}=22$.
10. $x^4-6x^2-3+4\sqrt{x^4-6x^2+9}=-7$.

Problems resulting in Quadratic Equations with One Unknown Quantity.

104. *Note* 1. A *rectangle* or *oblong* is a four-sided figure with two of its sides perpendicular to the other two. Its *diagonal* is a straight line drawn between two opposite angles. Its *area* = its *length* × its *breadth*.

1. The perimeter of a rectangular room (that is, its measure round about) is 64 feet; and 21 yards of carpet, 4 feet broad, are required to cover its floor. Find its dimensions.

Here, length + breadth = half of perimeter = 32 feet.

Let x = its length in feet $\therefore x(32-x)=63\times 4$ feet,
$\therefore 32-x$ = its breadth " $\therefore 32x-x^2=252$ feet,

$$\therefore x^2-32x=-252,$$
$$\therefore x^2-32x+16^2=4,$$
$$\therefore x=16\pm 2=18 \text{ or } 14 \text{ feet, its length;}$$
$$\text{and } 32-x=14 \text{ or } 18 \text{ feet, its breadth.}$$

Note 2. In a right-angled triangle, the base (B) and perpendicular (P) are the sides containing the right angle; the hypotenuse (H) is the side opposite it; and $B^2 + P^2 = H^2$.

2. Find the sides of a right-angled triangle whose base and hypotenuse are respectively 9 feet shorter and 9 feet longer than its perpendicular.

$$\left.\begin{array}{ll}\text{Let } x & = \text{the perpr. in feet,}\\ \text{then } x+9 & = \text{the hypot.} \quad ''\\ \text{and } x-9 & = \text{the base} \quad ''\end{array}\right\} \begin{array}{l}\therefore x^2 + (x-9)^2 = (x+9)^2,\\ \therefore x^2 + x^2 - 18x + 81,\\ \qquad = x^2 + 18x + 81,\end{array}$$

$$\therefore x^2 = 36x,$$

$\therefore x = 36$ feet, the perpendicular; $x + 9 = 45$ feet, the hypotenuse; $x - 9 = 27$ feet, the base.

3. Find the price of wheat when, if it were to fall 5s. per quarter, 1 bushel more would be got for £7, 10s.

$$x = \text{real price per qr.,} \quad \therefore \frac{150}{x} = \text{real number of qrs. got;}$$

$$x - 5 = \text{supposed} \quad '' \quad '', \quad \therefore \frac{150}{x-5} = \text{supposed} \quad '' \quad ;$$

$$\therefore \frac{150}{x-5} - \frac{150}{x} = \frac{1}{8}; \text{ whence } x^2 - 5x = 6000,$$

whence $x = 80$s. or -75s. per quarter, the real price,
and $x - 5 = 75$s. or -80s. $\quad ''\quad$, supposed $''$.

Note 3. The negative result, -75s., has a meaning. It indicates that 75s. will be the price per quarter if the conditions of the problem are reversed. That is, 75s. is the solution of the problem: 'Find the price of wheat when, if it were to *rise* 5s. per quarter, 1 bushel *less* of it would be got for £7, 10s.' Similarly, it will often be easily seen that other problems are capable of having their conditions altered so as to make the negative solution applicable. Compare Examples 5 and 6; and 1, where the double result implies that the terms length and breadth are interchangeable.

4. An officer, trying to arrange his men in a solid square, has 10 too many. If he arrange them in 7 lines fewer, each containing one-third more men, he will have 10 too few. How many men had he?

$$\left.\begin{array}{l}\text{Let (1) } x = \text{number of lines,}\\ \quad\therefore x = \text{number in a line;}\end{array}\right\} \therefore x^2 + 10 = \text{no. of men.}$$

$$\left.\begin{array}{l}\text{(2)} \quad x - 7 = \text{number of lines,}\\ \text{and } x + \dfrac{x}{3} = \dfrac{4x}{3} = \text{number in line;}\end{array}\right\} \therefore \frac{4x}{3}(x-7) - 10 = \text{〃 〃},$$

$$\therefore x^2 + 10 = \frac{4x}{3}(x-7) - 10,$$

whence $x = 30$; and $x^2 + 10 = 910$, the number of men he had.

5 Find the price of eggs per dozen, when one more in a shilling's worth lowers the price 2d. per dozen.

Let (1) x = real number for 1s.,

$$\therefore \frac{12}{x} = \text{real price each,} \quad \therefore \frac{144}{x} = \text{real price per dozen;}$$

(2) $x + 1$ = supposed number for 1s.;

$$\therefore \frac{12}{x+1} = \text{suppd. pr. each,} \quad \therefore \frac{144}{x+1} = \text{suppd. pr. per dozen;}$$

$$\therefore \frac{144}{x} - \frac{144}{x+1} = 2, \text{ whence } x^2 + x = 72,$$

whence $x = 8$ or -9, and $\dfrac{144}{x} = 18\text{d.}$ or -16d. the pr. per doz.

Note 4. The negative result indicates that 16d. is the solution of the problem: 'Find the price of eggs per dozen when 1 *fewer* in a shilling's worth *raises* the price 2d. per dozen.'

6. Find the buying-price of a horse on which, when sold for £24, as much per cent. is gained as it cost.

Let x = the buying-price,

$\therefore x$ = the gain per cent.,

$$\therefore \frac{x}{100} = \text{gain on £1,} \quad \therefore x \cdot \frac{x}{100} = \frac{x^2}{100} = \text{gain on £}x\text{;}$$

$$\therefore x + \frac{x^2}{100} = 24, \quad \therefore x^2 + 100x = 2400,$$

whence $x = £20$ or $-£120$.

Note 5. The negative result indicates that £120 is the solution of the problem: 'Find the buying-price of a horse on which, when sold for $-$ £24 (or when lost, and £24 is spent in vain in searching for him), as much per cent. is lost as he cost.'

EXERCISE LIV.

The following—namely, 1.(5), (6); 3.(2); 4.(1), (2); 5, 6, 12.(1), 18, 20, 23, 27, 32, 43, 45, 46.(2)—produce Pure Quadratics, and may be taken after Exercise XLVI.

1. Find two numbers (1) whose difference is 2, and product 624; (2) whose sum is 40, and product 351; (3) whose difference is 2, and their quotient equal to the less; (4) whose sum is 72, and their quotient one-sixth of the less; (5) whose quotient is 3, and product 972; (6) whose product is 884, and one of which is as much below 30 as the other is above 30.

2. Find two consecutive numbers (1) whose product is 756; (2) the sum of whose squares is 1301.

3. Find two consecutive odd numbers (1) whose product is 1 less than twice their sum; (2) whose product is 25 times the quotient of the greater by the less; (3) the sum of whose squares is to the sum of the numbers themselves as 17 to 4.

4. Find three consecutive numbers whose product equals (1) 24 times the middle one, (2) 16 times their sum, (3) 72 times the least, (4) the sum of whose squares is 22 times the least.

5. A cheese costs one-fourth as many pence per lb. as there are lbs. in it, and is sold for £1, 7s. Find its weight.

6. Find the sides of two squares when (1) one is $\frac{1}{4}$th the size of the other, and both together contain 5120 sq. yds.; (2) one is $\frac{2}{3}$rds the length of the other, and is 3920 sq. yds. less.

7. Find the sides of a field, square at first, which contains (1) 2496 sq. yds. when shortened 4 yds.; (2) 2156 sq. yds. when lengthened 15 ft.

8. A right-angled triangle has its base respectively 6 ft. longer and 6 ft. shorter than its other sides. Find its sides.

9. A rectangular field has one side 12 yds. shorter than the other, and its area is 1728 sq. yds. Find its diagonal.

10. The length of a room exceeds its breadth by 5 ft.; its diagonal is 25 ft. Find its area.

11. The perimeter of a room is 68 ft.; its area is 286·75 sq. ft. Find its sides.

12. A rectangle has a square cut off from one end of it. Find the sides of each part when (1) the remaining part is $\frac{1}{4}$th the length of the other, and the whole contains 720 sq. yds.; (2) the remaining part is 2 yds. more than $\frac{1}{3}$rd the length of the other, and the whole contains 1026 sq. yds.

13. A buys cloth for £3, 4s., sells it at 5s. 3d. per yd., and receives the cost price of 21 yds. How many yds. did he buy?

14. A person buys tea for £5, sells it at 6s. per lb., and gains what 4 lbs. cost him. How many lbs. did he buy?

15. An employer divides £15, 6s. equally among his men. Were each to get 1s. more, the number of men and the number of shillings given to each would be equal. Find what each gets.

16. Among a number of boys 224 marbles are equally divided. If each get 2 fewer, the number given to all will be the square of the number of boys. How many boys are there?

17. A boy trying to form a square with marbles in rows has 1 too few; arranging them with the no. in a row double the no. of rows, he has 3 over, but 2 rows fewer. How many has he?

18. A woman buys ducks and geese, 9 in all, for 35s. A goose costs her as many shillings as the number of her ducks, a duck one fewer than the number of her geese. What did each cost?

19. A party's railway fares amount to 32s. 6d.; as 4 do not pay, the others have to give 2s. 2d. more each. How many pay?

20. The height of a room is $\frac{2}{3}$rds of its breadth, its breadth is $\frac{3}{4}$ths of its length, and the area of its walls is 1792 sq. ft. Find its dimensions.

21. Two school-rooms have 600 pupils equally divided between them. In one 5 benches fewer are required, because 3 more can sit on a bench. How many benches are in each?

22. A lady buys 2 pieces of cloth for £5, 4s. each. The one is 8 yds. longer, but costs 1s. 6d. per yd. less than the other. How many yds. in each, and its price per yd.?

23. A, with 6d. too little to buy a fish that weighs thrice as many lbs. as it costs pence per lb., buys another $\frac{1}{3}$rd lighter and $\frac{1}{3}$rd dearer, and has 6d. over. How much money has he?

24. A's wage per day in shillings is to the number of days he works as 3 to 4; B's, who works 5 days longer, is to A's as 2 to 3. A earns £2, 10s. more than B. Find time and wage of each.

25. Find a number whose two digits differ by 2, its square with that of the number got by reversing the digits being 2340.

26. Find the price of tea when, were it to fall 8d. per lb., 4 oz. more of it would be got for a crown.

27. An officer has 16 men too few to form a solid square, but just enough to form a body containing one-sixth fewer lines with one-sixth more men in a line. How many men has he?

28. The frame of a picture is half its area, and everywhere 2 in. wide. Find the sides of the picture, which are as 4 to 5.

29. A, by going $\frac{1}{4}$ mile per hour faster than B, requires 1 hour less than he for 45 miles. Find the rate of each.

30. One kind of tea costs 2d. less per lb. than another, but 1 oz. more of it is got for 2s. 6d. Find price per lb. of each.

31. The distance of one end of a ladder from the foot of a wall is 8 ft. more than that of its other end, and its length is five-sevenths the sum of these distances. Find its length.

32. An oblong pond with sloping banks, and whose sides are as 5 to 6, has water run off till its length is $\frac{1}{6}$, its breadth $\frac{1}{10}$, and its surface 216 sq. yds. less than before. Find its sides.

33. The breadth of a room is $\frac{3}{4}$ths of its length. Another of equal size has its length 8 ft. greater, and its breadth 24 ft. less than the diagonal of the first. Find their dimensions.

34. A grazier buys oxen for £270, loses 2, sells the rest at gain of £2 per head, and loses £10 in all. How many did he buy?

35. A buys sheep for £50, sells all but 2 for £2 less, and, besides the value of these 2, gains 15s. per score. Find cost-price each.

36. A spends 1s. on peaches, 1s. on oranges, and gets 17 in all. Three oranges cost 1d. more than 2 peaches. Find price of each.

37. Were a train to go 3 furlongs per ho. slower, it would take 2 ho. 40 min. longer to run 210 miles. Find time it takes.

38. A rectangular carpet contains 396 sq. ft. If its width be lessened 6 ft, the remainder will be $\frac{1}{12}$th less than it would be were the length lessened as much. Find its dimensions.

39. A and B together earn £61. A works 4 wks. fewer, for 2s. a wk. more, and for 4s. in all more than B. Find the wage of each.

40. By selling a horse for (1) £39, (2) £56, (3) £65, 5s., as much per cent. is gained as it cost. Find the buying price.

41. By selling goods for (1) £16, (2) £21, (3) £22, 15s., as much per cent. is lost as they cost. Find the buying price.

42. By selling goods at a profit of (1) £5. (2) £8$\frac{1}{3}$, a gain per cent. equal to the selling price is made. Find prime cost.

43. I buy 48 sheep, sell $\frac{2}{3}$rds of them at a gain per cent. of as much as 3 cost, the rest at a gain per cent. of as much as 4 cost, and gain £10 in all. Find their cost price each.

44. Two sums, £800 in all, produce £31, 17s. 6d. of interest; each is 100 times the rate per cent. of the other; find them.

45. One pipe empties a cistern in 4 ho. more than another takes to fill it, and, both being open, 32 ho. more are required to fill it than the first takes to empty it. Find time of each and both.

46. Find time taken by each and by both to do a piece of work when (1) A takes 8 days less than B, and 1 day more than both; (2) A takes 1 ho. 12 min., B 32 min., more than both.

47. Find the price of eggs per dozen when (1) one more in a

shilling's worth lowers the price 2d. per dozen; (2) two fewer for 1s. 3d. raises the price 5d. per score.

48. A stationer, by giving 2 pens fewer for 1d. than he got for it, gains $\frac{1}{2}$d. per dozen. Find the buying price per gross.

49. One pipe empties a cistern in a certain time; it is filled by a second in 1 hour more, by a third in 1 hour more than by the second, and, with all open, in 4 times as long as by the second. Find the time required by each, and by all together.

50. A ladder placed so far from the foot of a wall reaches a height equal to $\frac{4}{5}$ths of its own length. Were it one-third longer it would reach 7 ft. higher. Find how high it reaches.

51. A family of 12 inherit an estate; $\frac{1}{4}$th of it is divided equally among the sons, the rest equally among the daughters. Find a son's share and a daughter's, which together make $\frac{1}{8}$th of it.

52. A's hoop makes 5 turns more than B's in 60 yds., but it would make only three more than B's in that distance if the rim of each measured 3 ft. more. Find their circumferences.

53. A buys 2 pieces of cloth with 23 coins of two kinds; for the first, which costs 4d. per yd. less than the second, he gives 2 crowns fewer than there are yds. in the second; for the second as many florins as there are yds. in the first. Price per yd. each?

54. To earn £7 A must work 7 days more than B, but only 5 days more if each earn 8d. a-day more. Find wage of each.

Simultaneous Quadratic Equations.

105. To Solve Quadratics containing Two Unknown Quantities.—There are several methods, the most useful of which are here given. Special attention should be paid to the first, second, and fourth.

I. By *Substitution.* (1) When one of the equations is of the First Degree (as in *Examples* 1 and 2), or (2) when such an Equation can be obtained from those given (as in *Example* 3): Find y in terms of x (or *vice versâ*) from the equation of the First Degree, and substitute this value for it in the other equation.

Examples.

1. (1) $x - y = 2, \quad \therefore y = x - 2,$
   (2) $xy = 15, \quad \therefore x(x-2) = 15; \quad \therefore x = 5, -3; \; y = 3, -5.$

2. $\left.\begin{aligned} (1)\quad 2x - 3y &= 3 \\ (2)\quad 3x^2 - 4xy &= 15 \end{aligned}\right\} \therefore y = \dfrac{2x-3}{3}, \quad \therefore 3x^2 - 4x \cdot \dfrac{2x-3}{3} = 15,$

$$\therefore x^2 + 12x = 45; \quad \therefore x = 3 \text{ or } -15, \quad y = 1 \text{ or } -11.$$

3. $\left.\begin{array}{l}(1)\ 2x^2+4x-3y=18, \quad \therefore 6x^2+12x-9y=54 \\ (2)\ 3x^2+5x-4y=26, \quad \therefore 6x^2+10x-8y=52\end{array}\right\}\begin{array}{l}\therefore 2x-y=2, \\ \therefore y=2x-2,\end{array}$

$\therefore 2x^2+4x-3(2x-2)=18; \quad \therefore x=3, \ -2; \ y=2x-2=4, \ -6.$

II. When each equation is homogeneous and of the Second Degree: Put $y=nx$, write the one expression over the other, and cancel out x^2.

An expression is homogeneous when the sum of the indices of one term is the same as that of the other terms.

Examples.

1. $\left.\begin{array}{l}(1)\ x^2+xy=4 \\ (2)\ y^2-xy=6\end{array}\right\}\therefore \dfrac{x^2+nx^2}{n^2x^2-nx^2}=\dfrac{1+n}{n^2-n}=\dfrac{4}{6}=\dfrac{2}{3}\left\{\begin{array}{l}\therefore 2n^2-5n=3, \\ \therefore n \quad =3, \ -\frac{1}{2};\end{array}\right.$

Let $n=\ \ 3, \ \therefore x^2+nx^2=4, \ \therefore 4x^2=4; \ \therefore x=\pm 1, \ y=nx=\pm 3;$

" $n=-\frac{1}{2}, \ \therefore x^2+nx^2=4, \ \therefore \frac{1}{2}x^2=4; \ \therefore x=\pm 2\sqrt{2}, \ y=\mp\sqrt{2}.$

2. $\left.\begin{array}{l}(1)\ x^2+xy+y^2=13 \\ (2)\ x^2-xy+y^2=\ \ 7\end{array}\right\}\therefore \dfrac{x^2+nx^2+n^2x^2}{x^2-nx^2+n^2x^2}=\dfrac{1+n+n^2}{1-n+n^2}=\dfrac{13}{7},$

$\therefore 13n^2-13n+13=7n^2+7n+7, \quad \therefore n=3 \text{ or } \frac{1}{3};$

Let $n=3, \quad \therefore x^2+3x^2+9x^2=13, \quad \because x=\pm 1, \quad \therefore y=nx=\pm 3;$

" $n=\frac{1}{3}, \quad \therefore x^2+\frac{1}{3}x^2+\frac{1}{9}x^2=13, \quad \therefore x=\pm 3, \quad \therefore y=nx=\pm 1.$

III. When both equations are symmetrical as to x and y, that is, when x and y can be interchanged without altering the expression: Put $x=m+n, \ \ y=m-n.$

Example.

$(1)\ x^4+y^4=82, \quad (2)\ x+y=4.$

$x+y=(m+n)+(m-n)=2m=4, \quad \therefore m=2;$

$\left.\begin{array}{l}x^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4 \\ y^4=m^4-4m^3n+6m^2n^2-4mn^3+n^4\end{array}\right\}\begin{array}{l}\therefore 2(m^4+6m^2n^2+n^4)=82, \\ \therefore \quad m^4+6m^2n^2+n^4 \ =41;\end{array}$

$\therefore n^4+24n^2=25, \ \therefore n=\pm 1; \ \therefore x=m+n=3, \ 1; \ y=m-n=1, \ 3.$

IV. By *Special Devices*, as in the following

Examples.

1. $\left.\begin{array}{l}(1)\ x-y=\ \ 6, \quad \therefore x^2-2xy+y^2=36 \\ (2)\ \ xy\ \ =16, \quad \therefore \qquad 4xy \qquad =64\end{array}\right\}\begin{array}{l}\therefore x^2+2xy+y^2=100, \\ \therefore x+y=\pm 10,\end{array}$

$\left.\begin{array}{l}\therefore \quad x+y=10 \text{ or } -10 \\ \text{But } x-y=\ \ 6 \text{ or } \quad 6\end{array}\right\}\begin{array}{l}\therefore 2x=16 \text{ or} -\ 4; \ \therefore x=8,-2; \\ \ \ 2y=\ 4 \text{ or} -16; \ \therefore y=2,-8.\end{array}$

2. $$\left.\begin{array}{l}(1)\ x^2+y^2=90\\(2)\quad xy\ \ =27\end{array}\right\}\begin{array}{l}\therefore\quad x^2+2xy+y^2=144,\quad \therefore x+y=\pm 12;\\ \text{and } x^2-2xy+y^2=\ \ 36,\quad \therefore x-y=\pm\ \ 6;\end{array}$$
$$\left.\begin{array}{l}\text{Let } x+y=12,\ +12,\ -12,\ -12\\ \text{and } x-y=\ \ 6,\ -\ \ 6,\ +\ \ 6,\ -\ \ 6\end{array}\right\}\begin{array}{l}\text{then } x=9,\ 3,\ -3,\ -9;\\ \qquad\ \ y=3,\ 9,\ -9,\ -3.\end{array}$$

3. $$\left.\begin{array}{l}(1)\ x+y=\ \ 8,\ \therefore x^2+2xy+y^2=64\\(2)\ x^2+y^2=34,\ \therefore x^2\quad+\quad y^2=34\end{array}\right\}\begin{array}{l}\therefore 2xy=30,\\ \therefore x^2-2xy+y^2=4,\end{array}$$
$\therefore x-y=\pm 2$; and (1) $x+y=8$; $\therefore x=5,\ 3$; $y=3,\ 5$.

4. $$\left.\begin{array}{l}(1)\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{4},\ \therefore \dfrac{1}{x^2}-\dfrac{2}{xy}+\dfrac{1}{y^2}=\dfrac{1}{16}\\(2)\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{16},\ \therefore \dfrac{1}{x^2}\quad+\quad\dfrac{1}{y^2}=\dfrac{5}{16}\end{array}\right\}\begin{array}{l}\therefore \dfrac{2}{xy}=\dfrac{4}{16},\\ \therefore \dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=\dfrac{9}{16},\end{array}$$
$$\left.\begin{array}{l}\therefore\quad \dfrac{1}{x}+\dfrac{1}{y}=\pm\dfrac{3}{4}\\ \text{But } \dfrac{1}{x}-\dfrac{1}{y}=\quad\dfrac{1}{4}\end{array}\right\}\begin{array}{l}\therefore \dfrac{2}{x}=1 \text{ or } -\dfrac{2}{4};\quad \therefore x=2,\ -4;\\ \dfrac{2}{y}=\dfrac{2}{4} \text{ or } -1;\quad \therefore y=4,\ -2.\end{array}$$

5. (1) $x^2-y^2=45$, (2) $x-y=3$. Divide (1) by (2) to get $x+y=15$.

6. (1) $x^2+xy=60$, (2) $y^2+xy=40$. Add them, get $x+y$, and divide each by it; $\therefore\ x=\pm 6,\ y=\pm 4$.

7. $$\left.\begin{array}{l}(1)\ x^3-y^3=98\\(2)\ x\ -y\ =\ \ 2\end{array}\right\}\left.\begin{array}{l}\therefore, \text{ dividing, } x^2+\ \ xy+y^2=49\\ \quad\text{from (2), } x^2-2xy+y^2=\ \ 4\end{array}\right\}\begin{array}{l}\therefore 3xy=45,\\ \therefore 4xy=60,\end{array}$$
$\therefore x^2-2xy+y^2=4,\ 4xy=60$; $\therefore\ x=5,\ -3$; $y=3,\ -5$; as in 1.

8. $$\left.\begin{array}{l}(1)\quad x^3+y^3\quad=65\\(2)\ x^2-xy+y^2=13\end{array}\right\}\left.\begin{array}{l}\text{dividing, } x+y=\ \ 5\\ \text{and (1), } x^3+y^3=65\end{array}\right\}\begin{array}{l}\therefore,\ \text{ as in 7,}\\ x=4,\ 1;\ y=1,\ 4.\end{array}$$

9. (1) $x^3+y^3=189$, (2) $x^2y+xy^2=180$. Add thrice (2) to (1); extract cube root, $\therefore x+y=9$; then as in 7, $x=5,\ 4$.

10. (1) $x^2+xy+y^2=13$, (2) $x^2-xy+y^2=7$. See 14.

11. (1) $x^2-y^2=5$, (2) $x^4+y^4=97$. Solve exactly as in 3.

12. $$\left.\begin{array}{l}(1)\ x^2+y^2-x+y=92\\(2)\qquad\quad xy\qquad\ =36\end{array}\right\}\begin{array}{l}\therefore x^2-2xy+y^2-x+y=20,\\ \quad\therefore (x-y)^2-(x-y)=20,\end{array}$$
$\therefore (x-y)^2-(x-y)+(\frac{1}{2})^2=\frac{81}{4}$, $\therefore x-y=5$ or -4,
$\therefore x-y=5,\ xy=36$; $\therefore$, as in 1, $x=9$ or -4, $y=4$ or -9.

13. $$\left.\begin{array}{l}(1)\ 3x^2-4xy=15,\ \ \therefore 9x^2-12xy=45\\(2)\ 2x\ -3y\ =\ \ 3,\ \ \therefore 8x^2-12xy=12x\end{array}\right\}\begin{array}{l}\therefore x^2+12x=45;\\ \text{then, as in I. 2.}\end{array}$$

14. (1) $x^4+x^2y^2+y^4=91$ ⎫ $\therefore$, by division, $x^2+xy+y^2=13$,
(2) $x^2-xy+y^2=7$ ⎭ and $x^2-xy+y^2=7$,
$\therefore$, by addn., $2(x^2+y^2)=20$, $\therefore x^2+y^2=10$ ⎫ $\therefore x=\pm3, \pm1$;
by subtrn., $2xy=6$, $2xy=6$ ⎭ $y=\pm1, \pm3$.

15. (1) $x^5-y^5=31$, (2) $x-y=1$;
$\therefore$, by division, $x^4+x^3y+x^2y^2+xy^3+y^4=31$,
$$\therefore x^4+y^4+xy(x^2+y^2)+x^2y^2=31 \qquad (3)$$
and from (2), $x^2-2xy+y^2=1$, $\therefore x^2+y^2=2xy+1$ (4)
$\therefore$, squaring (4), $x^4+2x^2y^2+y^4=4x^2y^2+4xy+1$
$$\therefore x^4+y^4=2x^2y^2+4xy+1 \qquad (5)$$
$\therefore 2x^2y^2+4xy+1+xy(2xy+1)+x^2y^2=31$, from (3), (4), and (5)
$\therefore 5x^2y^2+5xy=30$, $\therefore x^2y^2+xy=6$, $\therefore x^2y^2+xy+(\frac{1}{2})^2=\frac{25}{4}$,
$\therefore xy=2$ or -3, and $x-y=1$; $\therefore$ (as in 1) $x=2, -1$; $y=1, -2$.

Exercise LV.

1. $x-y=9$, $xy=112$; $x-3y=3$, $x^2+y^2=241$; $3x-2y=16$, $x^2-y^2=48$; $3x-4y=3$, $2x^2-3xy=5$.

2. $2x^2+3x-4y=53$, $4x^2+5x-9y=98$; $3x^2-2x+y=23$, $4x^2-3x+2y=31$.

3. $x^2+xy=8$, $y^2-xy=12$; $xy+x^2=12$, $2y^2-xy=24$; $x^2+xy=6$, $7xy-2y^2=12$; $4x^2-7xy+3y^2=2$, $3x^2-8xy+4y^2=3$.

4. $x^3-4xy-y^3=160$, $x-y=4$; $x^3-y^3=14xy$, $x-y=4$; $x^4+y^4=706$, $x-y=2$.

5. $x^2-y^2=6$, $x-y=2$; $\frac{1}{x^2}-\frac{1}{y^2}=16$, $\frac{1}{x}+\frac{1}{y}=2$; $\frac{1}{x^2}-\frac{1}{y^2}=\frac{1}{48}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{12}$.

6. $x^2+y^2=97$, $2xy=72$; $x^2+y^2=\frac{25}{36}$, $xy=\frac{1}{6}$; $x+y=17$, $xy=72$; $x+y=5$, $xy=5\frac{1}{4}$.

7. $x-y=10$, $xy=56$; $x+y=2\frac{1}{6}$, $xy=1$; $x^2+xy=15$, $xy+y^2=21$; $x^2-xy=17$, $xy-y^2=1$.

8. $x^2+y^2=53$, $x+y=9$; $x^2+y^2=89$, $x-y=3$; $x^4+y^4=82$, $x^2-y^2=8$; $x^4+16y^4=32$, $x^2-4y^2=0$.

9. $\frac{1}{x^2}+\frac{1}{y^2}=41$, $\frac{1}{x}-\frac{1}{y}=1$; $\frac{1}{x^2}+\frac{1}{y^2}=29$, $\frac{1}{x}+\frac{1}{y}=7$.

10. $x^2+xy+y^2=91$, $x^2-xy+y^2=31$; $x^2+2xy+4y^2=76$, $x^2-2xy+4y^2=28$; $x^4+x^2y^2+y^4=21$, $x^2+xy+y^2=7$.

11. $x^4-x^2y^2-2xy^3-y^4=7$, $x^2-xy-y^2=1$; $x^3+y^3=539$, $x+y=11$; $x^3-y^3=91$, $x-y=1$.

12. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=91,\ \dfrac{1}{x}+\dfrac{1}{y}=7;$ $\qquad \dfrac{1}{x^3}-\dfrac{1}{y^3}=61,\ \dfrac{1}{x}-\dfrac{1}{y}=1.$

13. $x^2-xy+y^2 = \frac{1}{2}(x^3+y^3)=76;$ $\qquad x^3+y^3=945,\ x^2y+xy^2=810;$ $\qquad x^3-8y^3=127,\ x^2y-2xy^2=21.$

14. $x^2+y^2+x+y=84,\ xy=24;$ $\qquad x^2+y^2-x+y=48,\ xy=14;$ $\qquad x^2-2xy=65,\ 3y-2x=22.$

15. $2x^2-3xy=36,\ 3x-4y=10;$ $\qquad 4x^2-9xy=28,\ 12y-5x=-8;$ $\qquad x^5-y^5=33,\ x-y=3.$

16. $\dfrac{6x}{y}+\dfrac{4y-4}{x+1}=12,\ y=11-x;$ $\qquad \dfrac{14-2x}{14-y}-\dfrac{5x}{2y}=\dfrac{5}{12},\ x=\dfrac{14-y}{4}.$

EXERCISE LVI.—PROBLEMS IN SIMULTANEOUS QUADRATICS.

1. Find two numbers (1) whose difference is 1, and the difference of their squares 23; (2) whose difference is 1, and product $8\frac{3}{4}$; (3) whose difference is 5, and the sum of their squares 97; (4) whose difference is 2, and the difference of their cubes 98; (5) the difference of whose cubes is 218, and their difference multiplied by their product 70; (6) whose sum multiplied by the greater gives $\frac{3}{8}$, and multiplied by the less, $\frac{3}{16}$.

2. Find two numbers such that the sum of their squares is 360, and the first is to the second as the second is to 54.

3. Find two numbers whose sum multiplied by the greater gives 48, and whose difference multiplied by the less gives 8.

4. Find two numbers whose sum multiplied by the greater gives 40, and the product of whose sum and difference is 16.

5. Find two numbers whose sum multiplied by the greater is 12 times the less, and, multiplied by the less, is thrice the greater.

6. A ladder 20 ft. shorter than the breadth of a street, is placed 5 ft. from the middle of the street, and its top reaches 10 ft. higher on one side of the street than the other. Find its length.

7. A rectangular pond with sloping banks, has water run into it till its length is increased by $\frac{1}{16}$, its breadth by $\frac{1}{8}$, its perimeter by 8 yds., and its area by 76 sq yds. Find its original area.

8. A farmer spends £1152 on horses and cows, buying as many horses as he gives pounds for a cow, and as many cows as he gives pounds for a horse. He buys more cows than horses, and he sells them all at the difference between the cost-price of a horse and a cow, and loses £112. How many of each did he buy?

9. A woman for £3, 4s. buys ducks and geese at as many shillings each as there are geese; all her geese and as many ducks die, and she sells the rest at half as many shillings each

as the number of ducks she bought, and loses 25 per cent. How many of each did she buy?

10. A person bought 28 yds. of cloth, in two pieces, which differed in price by 4d. a yard. The dearer was bought with half-crowns and the cheaper with florins, and there were as many yds. of the cheaper more than of the dearer as there were florins more than half-crowns. The total number of coins being 24, how much money was spent?

EQUATIONS CONTAINING SURDS.

106. RULE. (1) Collect the quantities so as to have only one surd on one side; (2) raise both sides to the power indicated by the surd index; (3) repeat the process if a surd be still left, unless the surd left be the square root of the other unknown, in which case proceed as in Exercise LIV.

Example.

$3(\sqrt{x}-2)(\sqrt{x}-1)=6-\sqrt{x}, \quad \therefore 3(x-3\sqrt{x}+2)=6-\sqrt{x},$

$\therefore 3x-9\sqrt{x}+6=6-\sqrt{x}, \quad \therefore 3x=8\sqrt{x}, \quad \therefore 9x^2=64x, \quad \therefore x=\frac{64}{9}, 0.$

EXERCISE LVII.

Simple Equations, 1—7; *Quadratics*, 8—15.

1. $\sqrt{x}=3$; $2\sqrt{x}=4$; $\sqrt[3]{x}=5$; $\sqrt[4]{5x}=10$; $x^{\frac{1}{5}}=6$; $2x^{\frac{1}{3}}=8$.
2. $x^{\frac{2}{3}}=4$; $2x^{\frac{3}{2}}=16$; $\sqrt{x+5}=4$; $\sqrt{2x-1}=5$; $\sqrt[3]{7-3x}=4$.
3. $3(x-3)^{\frac{1}{2}}=15$; $2(x-1)^{\frac{3}{2}}=54$; $5+2\sqrt{x}=9$; $5-\sqrt{2x-1}=2$.
4. $\sqrt{x+8}+\sqrt{x}=4$; $\sqrt{x-11}+1=\sqrt{x}$; $\sqrt{x+7}-\sqrt{x-5}=2$.
5. $(\sqrt{x}+1)(\sqrt{x}-1)=8$; $\quad \sqrt{2x-3}=\sqrt{5}-\sqrt{2x}$.
6. $\sqrt{x}+\dfrac{6}{\sqrt{x}}=\sqrt{x+13}$; $\quad \sqrt{x}-\dfrac{3}{\sqrt{x-7}}=\sqrt{x-7}$.
7. $\dfrac{\sqrt{x}-2}{\sqrt{x}-5}=\dfrac{\sqrt{x}-4}{\sqrt{x}-6}$; $\quad \dfrac{1}{\sqrt{x-4}}-\dfrac{1}{\sqrt{x+4}}=\dfrac{2}{\sqrt{x^2-16}}$.
8. $x-4\sqrt{x}=5$; $\quad 3x-8\sqrt{x}=60$; $\quad 2x-\sqrt{4x-x^2}=1$.
9. $6x-\sqrt{7-6x}=1$; $x+5=3\sqrt{7-3x}$; $\sqrt{7x+9}+1=3\sqrt{x}$.
10. $\sqrt{3x+1}=\sqrt{2x-7}+2$; $\quad 2\sqrt{x}-\sqrt{3x+6}=1$.
11. $5\sqrt{x}-2\sqrt{6x-5}=1$; $\quad \sqrt{8(x-1)}-2\sqrt{x}=x$.
12. $(2+\sqrt{x})(5-\sqrt{x})=6\sqrt{x}$; $\quad \sqrt{10+x}-\sqrt{10-x}=2$.
13. $(\sqrt{x}+7)(2\sqrt{x}-3)=5\sqrt{x}-1$; $\quad \sqrt{17+x}=8-\sqrt{17-x}$.
14. $x^2+2\sqrt{x^2+x+4}=20-x$; $\quad \sqrt{3x-5}+\sqrt{3x+4}=\sqrt{11x+4}$.
15. $5x-2\sqrt{x^2+5x+2}=6-x^2$; $\quad \sqrt{4x+5}-\sqrt{3x+1}=\sqrt{x^2-4}$.

Equations with Literal Coefficients.

107. Simple equations of this kind in one unknown quantity have already been given; a few simultaneous equations and quadratic equations are now added.

Examples.

1. Solve $ax + by = bx - ay = c$,

$$\left.\begin{aligned} abx + b^2y &= bc \\ abx - a^2y &= ac \end{aligned}\right\} \therefore y(a^2+b^2) = c(b-a), \quad \therefore y = \frac{-c(a-b)}{a^2+b^2},$$

and $$\left.\begin{aligned} a^2x + aby &= ac \\ b^2x - aby &= bc \end{aligned}\right\} \therefore x(a^2+b^2) = c(a+b), \quad \therefore x = \frac{c(a+b)}{a^2+b^2}.$$

2. Solve $x^2 + nx = 6n^2$,

Completing the square, $x^2 + nx + \left(\frac{n}{2}\right)^2 = 6n^2 + \frac{n^2}{4} = \frac{25n^2}{4}$,

$$\therefore x + \frac{n}{2} = \pm\frac{5n}{2}, \qquad \therefore x = 2n \text{ or } -3n.$$

Exercise LVIII.

1. $x + y = 2a$, $x - y = 2b$; $\quad bx + cy = de$, $x + y = a$; $\quad lx + my = n$, $mx + ny = l$.

2. $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{3a} + \frac{y}{6b} = \frac{2}{3}$; $\quad \frac{a}{x} + \frac{b}{y} = c$, $\frac{a}{x} - \frac{b}{y} = d$; $\quad \frac{3}{mx} + \frac{4}{ny} = 7$, $\frac{4}{mx} - \frac{3}{ny} = 1$.

3. $(a+b)x + (a-b)y = 2ac$, $(b+c)x + (b-c)y = 2bc$; $\quad (a+b)x - ay = 2ab$, $cx - (b+c)y = b^2 + bc - ab$.

4. $ax - by = bx - ay = c$; $\quad mx + ny = 1 = nx - my$.

5. $x + y = a$, $x^2 + y^2 = b^2$; $\quad xy = a^2$, $x + y = 2b$.

6. $x^2 + 8a^2 = 6ax$; $\quad x^2 - a^2 = ab - bc$; $\quad x^2 + 2(a-b)x = 2ab - b^2$.

7. $(a+x)(a-x) = b(2a+b) - 2(x+a)(x-a)$.

8. $\frac{x}{a} - \frac{a^2b}{x} = \frac{ab^2}{x} - \frac{x}{b}$; $\quad \frac{x^2}{a^2} + \frac{a^2}{x^2} = \frac{x^2}{b^2} + \frac{b^2}{x^2}$; $\quad \frac{2a}{a-x} + \frac{a-x}{a+x} = \frac{7a^2}{a^2-x^2}$.

9. $\frac{x}{a} - \frac{a+x}{ax} = \frac{a}{x}$; $\quad \frac{x}{x-a} + \frac{x}{x+a} = \frac{3x}{4a}$; $\quad \frac{x^2}{a-b} - \frac{2x}{a+b} = \frac{1}{b-a}$.

10. $\frac{x}{a} - \frac{a}{x} = \frac{2x}{3a} + \frac{2a}{x}$; $\quad \frac{1}{x-m} + \frac{1}{x-n} = \frac{1}{m} + \frac{1}{n}$; $\quad \frac{m}{x-m} - \frac{m}{n} = \frac{n}{m} - \frac{n}{x-n}$.

11. $\sqrt{x+2a} + \sqrt{x+2b} = \sqrt{2(x+a+b)}$; $\quad \sqrt{x+3a^2} + \sqrt{x} = \sqrt{x+8a^2}$.

12. $2x(x + \sqrt{n+x^2}) = n(n-1)$; $\quad \sqrt{mx+n^2} - m = n + \sqrt{m^2+nx}$.

TEST EXERCISES.

NOTE.—Exercises XI.–XX. afford revisal of Parts I. II. and III.

I.

1. Simplify $(a+b)^4 - 2(a^2-b^2)^2 + (a-b)^4$.
2. Multiply $a^{\frac{4}{3}} - a^{\frac{2}{3}} + a^{-\frac{2}{3}} - a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}} + a^{\frac{1}{3}} - a^{-\frac{1}{3}} - a^{-\frac{2}{3}}$.
3. Arrange in order of magnitude
   (1) $3\sqrt{7}$, $4\sqrt{5}$, $\frac{5}{2}\sqrt{8}$, $\frac{2}{3}\sqrt{135}$.
   (2) $\sqrt{15}$, $2\sqrt[3]{5}$, $2\sqrt[6]{30}$.
4. Solve (1) $x^6 - 9x^3 + 8 = 0$.
   (2) $x^2 - 2xy + y^2 = 9$; $x^3 - y^3 = 279$.
5. A and B, working together, can do a piece of work in $5\frac{5}{8}$ days; working separately, B takes 6 days longer than A. Find in how many days each can do the work.

II.

1. Simplify $\{a^{-\frac{1}{4}}(a^{-1}b^2\sqrt[4]{ab})^{-\frac{1}{3}}b^{\frac{3}{4}}\}^{-1}$.
2. If $x = \sqrt{\dfrac{1-a}{1+a}}$, then $\dfrac{1+x}{1-x} = \dfrac{1+\sqrt{1-a^2}}{a}$.
3. Find the square root of
   (1) $1 - 6a + 15a^2 - 20a^3 + 15a^4 - 6a^5 + a^6$.
   (2) $x^2 + 2x - 1 - 2x^{-1} + x^{-2}$.
4. Solve (1) $\dfrac{x-3}{x-5} - \dfrac{x-5}{x-6} = \dfrac{1}{6}$.
   (2) $x^2 + xy = 63$; $x^2 - y^2 = 77$.
5. A number of two digits is equal to $1\frac{1}{2}$ times the product of the digits, and the digit in the unit's place is double the digit in the ten's place. Find the number.

III.

1. Simplify (1) $\dfrac{(2^3)^2 \cdot (3^2)^6 \cdot (4^2)^3}{(6^2)^4 \cdot (2^5)^3 \cdot (3^2)^2}$. (2) $\dfrac{(x^{p+2q})^3 \cdot (x^{q+2r})^3 \cdot (x^{r+2p})^3}{(x^p \times x^q \times x^r)^9}$.
2. Show that $\dfrac{(2a-b-c)^2 + (2b-c-a)^2 + (2c-a-b)^2}{(b-c)^2 + (c-a)^2 + (a-b)^2} = 3$.
3. Simplify (1) $\dfrac{9+\sqrt{45}}{9-\sqrt{45}} + \dfrac{9-\sqrt{45}}{9+\sqrt{45}}$. (2) $\dfrac{(3+2\sqrt{2})^2}{3-2\sqrt{2}}$.
4. Solve the following equations by resolution into factors:
   (1) $x^2 - 3x - 70 = 0$. (2) $x^3 + 5x^2 + 4x = 0$.
   (3) $5x^2 + 50 = 35x$. (4) $2x^2 + 5x = 3$.

5. The sum of the cubes of two numbers is 189, and the sum of their squares diminished by their product is 21. Find the numbers.

IV.

1. Square (1) $1 + x + \sqrt{1 - x}$. (2) $\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}$.
2. Find the square root of

   (1) $a^4 + a^3 + \frac{11}{12}a^2 + \frac{1}{3}a + \frac{1}{9}$.

   (2) $16(x^4 + 1) + 41x^2 - 24x(x^2 + 1)$.
3. Simplify $\dfrac{\sqrt{x + 2y} + \sqrt{x - 2y}}{\sqrt{x + 2y} - \sqrt{x - 2y}} + \dfrac{\sqrt{x + 2y} - \sqrt{x - 2y}}{\sqrt{x + 2y} + \sqrt{x - 2y}}$.
4. Solve (1) $\dfrac{1}{x + 1} + \dfrac{1}{x - 1} + \dfrac{1}{x + 7} + \dfrac{1}{x - 7} = 0$.

   (2) $x^2y^2 + 6xy = 72$; $x + y = 5$.
5. The sum of the squares of two consecutive even numbers is 1252. Find the numbers.

V.

1. Square $a^3 - 2a + 2a^{-1} - a^{-3}$, and divide $a^{\frac{3}{2}} - a^{-\frac{3}{2}}$ by $a^{\frac{1}{2}} - a^{-\frac{1}{2}}$.
2. Evaluate $\dfrac{2(x - y)(1 - xy)(x + 1)(y + 1}{(x^3 + 1)(y^3 + 1)}$, when $x = \sqrt{2} - \sqrt{3}$. and $y = \sqrt{2} + \sqrt{3}$.
3. Find the square root of

   (1) $(x + 1)(x + 2)(x + 3)(x + 4) + 1$. (2) $32 + 10\sqrt{7}$.
4. Solve (1) $2x + \sqrt{2x + 11} = 19$.

   (2) $x^2 - 9y^2 = 24$; $x + 3y = 12$.
5. In walking a mile, A takes 192 steps more than B; if each were to increase his step by 3 inches, A would take only 160 steps more than B. Find the pace of each.

VI.

1. Find by inspection

   $(x - 3y + 5z)^2$; $(2a - 3b)^3$; $(m + n)^4$.
2. Simplify (1) $\sqrt{20} + \sqrt{45} + \sqrt{\frac{4}{5}}$. (2) $(2\sqrt{3} + 5\sqrt{2})(5\sqrt{3} - 2\sqrt{2})$.

   (3) $\sqrt[3]{56} + \sqrt[3]{27} - \sqrt[3]{189}$. (4) $(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})$.
3. Find the square root of

   (1) $9x^2 + 6xy - 12x + y^2 - 4y + 4$. (2) $2p + 3 + 2\sqrt{p^2 + 3p + 2}$.
4. Solve (1) $x^2 + 4\sqrt{x^2 + x + 5} = 40 - x$.

   (2) $x^3 + y^3 = 35$; $x^2y + xy^2 = 30$.

5. The area of a rectangular field is one acre, and the length is one yard more than 3 times the breadth. Find its dimensions.

VII.

1. Write down the following quotients:

$$(1)\ \frac{a-b}{\sqrt[3]{a}-\sqrt[3]{b}}.\quad (2)\ \frac{x^{\frac{3}{2}}+y^{\frac{3}{2}}}{\sqrt{x}+\sqrt{y}}.\quad (3)\ \frac{x-2\sqrt{x}+1}{\sqrt{x}-1}.$$

2. Simplify (1) $(a^2+b^2)^2(a+b)^2(a-b)^2$.

(2) $\{(x+1)(x^2-x+1)\}^2+\{(x-1)(x^2+x+1)\}^2$.

3. Find the square root of $3(a+b)+2\sqrt{2(a^2+b^2)+5ab}$.

4. Solve (1) $\dfrac{x+3}{x}+7\cdot\dfrac{x}{x+3}=\dfrac{23}{4}$.

(2) $2x+3y=5;\quad 2x^2+5xy+3y^2=10$.

5. Two men walk from Edinburgh to Glasgow, a distance of 42 miles. By walking half a mile an hour faster the one accomplishes the journey in two hours less than the other. What is the rate of each?

VIII.

1. Square (1) $\{(x^2+y^2)^{\frac{1}{2}}-(x^2-y^2)^{\frac{1}{2}}\}$. (2) $(\sqrt{x}+\sqrt{y}-\sqrt{z})$.

2. Simplify $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}+\dfrac{2\sqrt{3}}{3}$, and $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}-\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

3. Find the square root of $4a^2+b^2+9c^2-4ab-6bc+12ac$.

4. Solve (1) $\sqrt{x}+\sqrt{x+5}=\dfrac{15}{\sqrt{x+5}}$.

(2) $x^2+y^2-48=x-y=5$.

5. A rectangular field has an area of 12,000 square yards, and its boundary wall measures 460 yards. Find its length and breadth.

IX.

1. Find the values of $256^{\frac{1}{4}}$, $25^{-\frac{1}{2}}$, $16^{\frac{3}{4}}$, $27^{\frac{2}{3}}$, $144^{-\frac{3}{2}}$.

2. Evaluate the following expression:

$$(a+b+c)^3+(a+b-c)^3+(a-b+c)^3+(-a+b+c)^3.$$

3. Simplify (1) $\dfrac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$. (2) $\dfrac{6\sqrt{5}+8\sqrt{7}}{\sqrt{157+24\sqrt{35}}}$.

4. Solve (1) $9\left(x^2+\dfrac{1}{x^2}\right)=82$.

(2) $x^2+y^2+2x+2y=56;\quad xy-x-y=4$.

5. The sum of two numbers is 9, and the sum of their reciprocals is $\frac{1}{2}$. Find the numbers.

X.

1. Simplify $\left(\dfrac{a}{b^{\frac{1}{3}}}\right)^{\frac{1}{2}} \times \left(\dfrac{a^{-1}}{b^{-\frac{1}{3}}}\right)^{\frac{1}{3}} \times (\sqrt{a^{\frac{1}{3}}b^{\frac{1}{5}}})^{-2} \times (\sqrt{a^{-1}b^{-1}})^{-\frac{1}{3}} \times \sqrt[3]{ab}$.

2. If $x = 3 - \sqrt{3}$, show that $x^2 + \dfrac{36}{x^2} = 24$.

3. Find the square root of

   (1) $\dfrac{4x^2}{z^2} + \dfrac{z^2}{x^2} + \dfrac{9y^2}{z^2} + 4 - \dfrac{6y}{x} - \dfrac{12xy}{z^2}$. (2) $8 - 2\sqrt{15}$.

4. Solve (1) $(x^2 - 7x)^2 + 22(x^2 - 7x) = -120$.

   (2) $x^2 + xy = 28$; $xy - y^2 = 3$.

5. A farmer bought a certain number of sheep for 50 guineas. Had he paid 5 shillings less for each sheep he could have bought five more. How many did he buy, and at what price?

XI.

1. If $a = 2b$, $b = 3c$, $c = 4d$, find the value of the square root of $3a + 5b - 6c + 20d$ when $d = 2$.

2. Resolve into factors

   $x^4 - 256$; $x^2 - 3\frac{3}{4}x - 1$; $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24$.

3. Reduce to their simplest form

   (1) $\dfrac{a^3b^3 + c^3d^3}{a^2b^2 - c^2d^2}$. (2) $\dfrac{x^2 + y^2 + z^2 + 2(yz + zx + xy)}{x^2 - (y^2 + z^2 + 2yz)}$.

4. Solve (1) $\dfrac{x^2 - 16}{x + 4} + \dfrac{x^2 - 9}{x - 3} = 11 + \dfrac{x^2 - 4}{x + 2}$.

   (2) $4x + 6y - 21 = 0 = 6x + 8y - 43$.

   (3) $(m^2 - n^2)x^2 + 2(m^2 + n^2)x = n^2 - m^2$.

5. Simplify $(x^2 - \sqrt{2} \cdot ax + a^2)^2(x^2 + \sqrt{2}\, ax + a^2)^2$.

6. When 4 is added to the numerator of a certain fraction, it becomes $\frac{3}{4}$, and when 2 is subtracted from its denominator, it becomes $\frac{1}{2}$. Find the fraction.

XII.

1. Collect the coefficients of powers of x in the following:

   (1) $(x-a)^3-(x-b)^3$. (2) $(ax^2-bx+c)^2-(mx^2-nx+r)^2$.

2. Prove the following identity:

   $(a + b - c)^2(b + c - a)^2(c + a - b)^2 = \{a^2 - (b - c)^2\}$
   $\{b^2 - (c - a)^2\}\{c^2 - (a - b)^2\}$.

3. Divide $\dfrac{x}{y} - \dfrac{y}{x} + \dfrac{y}{z} - \dfrac{z}{y} + \dfrac{z}{x} - \dfrac{x}{z}$ by $(x - y)(y - z)(z - x)$.

4. Solve (1) $(3x+4)^2-(2x+3)^2=x(5x+7)$.

(2) $4x^2+y^2=61$; $xy=15$.

(3) $x+3x^{\frac{1}{2}}=10$.

5. Find the square root of $x^4+\frac{x^2}{4}+\frac{4}{x^2}-x^3+4x-2$.

6. A nurseryman planted a certain number of young trees in the form of a square, and had 9 trees over. Had he put one tree more in the side of the square he would have been 34 trees short. Find the number of trees.

XIII.

1. Simplify the expression: $-7ac-\{2c(a-3b)-3a(5c-2b)\}$, and find its value when $c=\frac{ab}{a+b}$.

2. Resolve into factors

x^3-3x^2-88x; $x^2+y^2+2(xy+xz+yz)$; $a^2+2ab-9c^2-6bc$.

3. Simplify $$\frac{\left(1+\frac{c}{a+b}+\frac{c^2}{(a+b)^2}\right)\left(1-\frac{c^2}{(a+b)^2}\right)}{\left(1-\frac{c^3}{(a+b)^3}\right)\left(1+\frac{c}{a+b}\right)}.$$

4. Solve (1) $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$.

(2) $2x^2-xy=4$; $8y^2-7xy=4$.

(3) $\sqrt{a+x}+\sqrt{a-x}=\sqrt{b}$.

5. Find the continued product of $3\sqrt{2}-4\sqrt{3}$, $\sqrt{5}+2$, $\sqrt{5}-2$, $2+\sqrt{7}$, $3\sqrt{2}+4\sqrt{3}$, and $2-\sqrt{7}$.

6. The sum of the squares of two numbers is 89, and the product of the numbers is 40. Find the numbers.

XIV.

1. Divide $(3a+2b+c)^2-(a+2b+3c)^2$ by $4(a+b+c)$, and verify the result when $a=3$, $b=2$, $c=1$.

2. Find the G.C.M. of $(x^2+y^2)^2-x^2y^2$ and $(x+y)^4-2xy(x+y)^2+x^2y^2$.

3. Simplify $\left(\frac{x-a}{ax+1}-\frac{x-b}{bx+1}\right)\div\left(\frac{1}{ax^2+x}-\frac{1}{bx^2+x}\right)$.

4. Solve (1) $\cdot25x+\frac{\cdot5x-\cdot25}{1\cdot875}=1\cdot5+\frac{2\cdot5x}{8}$.

(2) $xy-2x=21$, $xy+3y=50$.

(3) $\frac{x-1}{\sqrt{x}+1}-1=\frac{\sqrt{x}-1}{2}$.

5. Find the square root of $x^{\frac{2}{3}} - 4x^{\frac{1}{3}}y^{\frac{1}{3}} + 4y^{\frac{2}{3}} + 6x^{\frac{1}{3}}z^{\frac{1}{3}} - 12y^{\frac{1}{3}}z^{\frac{1}{3}} + 9z^{\frac{2}{3}}$.

6. By selling a horse for £75, a horse-dealer gained as much per cent. as the horse cost him. Find the cost price of the horse.

XV.

1. Find the value of

$$\frac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 - c^2 + 2bc} \text{ when } a = \tfrac{1}{2},\ b = 1,\ c = -\tfrac{3}{2}.$$

2. Resolve into factors

$$2mn - m^2 - n^2 + 1; \quad a^4 - 10a^2 + 9; \quad x^3 - y^3 + xy(x - y).$$

3. Simplify $\dfrac{x^3 + y^3}{x + y} + \dfrac{x^3 - y^3}{x - y} - \left(\dfrac{x}{y} + \dfrac{y}{x}\right)\left(\dfrac{x^3 - y^3}{x - y} - \dfrac{x^3 + y^3}{x + y}\right)$.

4. Solve (1) $\dfrac{x + 1\frac{1}{2}}{3} + \dfrac{2x + 5}{2\frac{1}{2}} = 5 + \dfrac{3x}{3\frac{1}{4}}$.

(2) $\dfrac{1}{x + 1} - \dfrac{2}{x + 2} = \dfrac{3}{x + 3} - \dfrac{4}{x + 4}$.

(3) $xy = 35; \quad yz = 15; \quad zx = 21.$

5. Simplify (1) $\dfrac{2^4}{2(2^2)^2}$. (2) $\left(\dfrac{x^m}{x^n}\right)^{m+n} \times \left(\dfrac{x^n}{x^l}\right)^{n+l} \times \left(\dfrac{x^l}{x^m}\right)^{l+m}$.

6. A rectangular garden measures 1000 square yards. If the length be diminished by 4 yards, and the breadth increased by 5 yards, the area will be increased by 80 square yards. Find its dimensions.

XVI.

1. Divide $(a + 2)^4 + 4(a + 2)^3 + 6(a + 2)^2 + 4(a + 2) + 1$ by $a + 3$.

2. If $x = \dfrac{p + q}{p - q}$ and $y = \dfrac{p - q}{p + q}$, prove $\dfrac{x + y}{x - y} = \dfrac{p}{2q} + \dfrac{q}{2p}$.

3. Simplify (1) $\dfrac{(x + a)^2 - (x - b)^2}{2x + (a - b)}$. (2) $\dfrac{x^2 + \left(a + \frac{1}{a}\right)xy + y^2}{x^2 - \left(a - \frac{1}{a}\right)xy - y^2}$.

4. Solve (1) $(x + 1)^2 - 3(x - 2)^2 = 4(x + 3)^2 - (6x^2 + 77)$.

(2) $\dfrac{\sqrt{x + a} + \sqrt{x - a}}{\sqrt{x + a} - \sqrt{x - a}} = b$. (3) $\dfrac{4}{x + 7} + \dfrac{12}{(x + 7)^2} = 1$.

5. Simplify $(\sqrt{x + y + z} + \sqrt{x - y + z})^2 \times (\sqrt{x + y + z} - \sqrt{x - y + z})^2$.

6. Find a number whose square increased by 16 is equal to 20 times the excess of the number over 4.

XVII.

1. If $a = \frac{3}{4}b = 6$, and $x = \frac{2}{3}y = -6$, find the value of $(a - b)^2 - 2(a - b)(x - y) + (x - y)^2$.

2. Resolve into factors

$$a^2x^6 - a^8; \quad x^3 + 6x^2 - x - 30; \quad m^2 + 1 + \frac{1}{m^2}.$$

3. Simplify $\dfrac{\{(ax + by)^2 + (ay - bx)^2\}\{(ax + by)^2 - (ay + bx)^2\}}{a^2x^2 + b^2x^2 - a^2y^2 - b^2y^2}$.

4. Solve (1) $\dfrac{x^3 + a^3}{x + a} + 3a(a - x) = \dfrac{x^3 - a^3}{x - a} - 2a^2$.

(2) $\dfrac{x + 2y}{5z + 1} = 1; \quad \dfrac{5y + 4z}{3x + 1} = 2; \quad \dfrac{z + 4x}{y + 1} = 3.$

(3) $x^2 - 2(a^2 + b^2)x + a^4 + b^4 = 2a^2b^2$.

5. Find the continued product of

$$(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})(x + 3 + \sqrt{5})(x + 3 - \sqrt{5}).$$

6. After paying to A £1 more than half my money, to B £1 more than one-third of the remainder, and to C £2 more than one-fourth of what then remained, I had £16 15s. left. How much money had I at first?

XVIII.

1. Find the value of $\dfrac{m - n}{1 + mn}$, when $m = \dfrac{x + y}{x - y}$, $n = \dfrac{y}{x}$.

2. Simplify $\dfrac{a^2 - 3ab + 2b^2}{a - 2b} + \dfrac{a^2 - ab - 6b^2}{a - 3b} - \dfrac{a^2 - 7ab + 12b^2}{a - 4b}$.

3. Find the sum of

$$\frac{x}{x^2 - 1} + \frac{x^2 + x - 1}{x^3 - x^2 + x - 1} + \frac{x^2 - x - 1}{x^3 + x^2 + x + 1} - \frac{x^3}{x^4 - 1}.$$

4. Solve (1) $\dfrac{x + a}{x - b} = \dfrac{x + c}{x - d}$. (2) $\dfrac{72x}{6 + x} = 12x - 1$.

(3) $x^2 + xy + y^2 = 31; \quad x^2 - xy + y^2 = 21$.

5. Simplify (1) $\{(a^{\frac{1}{2}}b^{-\frac{1}{3}}\sqrt[6]{ab})^{\frac{3}{2}} \cdot a^2b^2\}^{-\frac{1}{4}}$. (2) $\sqrt{4 + \sqrt{7}}$.

6. A bag contains 180 gold and silver coins of the total value of £60. Each gold coin is worth as many pence as there are silver coins, and each silver coin as many pence as there are gold coins. How many are there of each kind?

XIX.

1. When $a = 6\cdot25$ and $b = 3\cdot75$, find the values of
$$(1)\ \frac{(a-b)^2}{a^2-b^2}.\quad (2)\ \frac{a^2+ab}{(a+b)^2}.\quad (3)\ \frac{a^2-ab+b^2}{a^3+b^3}.$$

2. Find the G.C.M. and the L.C.M. of
$x^3 - (a+3)x^2 + (3a+2)x - 2a$ and
$x^3 + (a+1)x^2 + (a-2)x - 2a$.

3. Collect $\dfrac{bc(a+d)}{(a-b)(a-c)} - \dfrac{ac(b+d)}{(a-b)(b-c)} + \dfrac{ab(c+d)}{(a-c)(b-c)}$.

4. Solve (1) $\dfrac{2\cdot5(3\cdot1x+2)}{4x} = 3\cdot1875$. (2) $\dfrac{1}{x^2-3x} + \dfrac{1}{x^2+4x} = \dfrac{9}{8x}$.
(3) $(x+2)(y+3) = 35$; $(x+2)^2 + (y+3)^2 = 74$.

5. Find to three places of decimals the values of
$$\frac{\sqrt{10}+3}{\sqrt{10}-3};\quad \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}};\quad \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

6. A carriage and pair costs £150. Five times the price of one horse equals seven times the price of the other, and the carriage costs one-fourth as much as the pair of horses. Find the price of the carriage and of each horse.

XX.

1. Expand and arrange according to powers of x
$\{(x+1)^6 + (x+1)^4 + (x+1)^2 + 1\} \times \{(x+1)^2 - 1\}$.

2. Divide the sum of $4x(x^2 + xy + \frac{1}{2}y^2)$, $3y(x^2 - xy - \frac{1}{2}y^2)$, $2x(2x^2 - xy + 2y^2)$, and $y(13x^2 + 8xy + 7y^2)$, by $2x + 3y$.

3. Simplify $\left(\dfrac{a}{b} - \dfrac{b}{a} + \dfrac{c}{d} - \dfrac{d}{c}\right)\left(\dfrac{a}{b} - \dfrac{b}{a} - \dfrac{c}{d} + \dfrac{d}{c}\right)$.

4. Solve (1) $\dfrac{x-1}{x-2} - \dfrac{x-2}{x-5} = \dfrac{x-2}{x-5} - \dfrac{x-2}{x-3}$.
(2) $x^4 + y^4 = 272$; $x + y = 6$.
(3) $\sqrt{12x-35} + 2\sqrt{2x-1} = \sqrt{20x+21}$.

5. Find the square root of
$(a^2 - 2a + 1)(4a^2 + 12a + 9)(16a^2 - 40a + 25)$.

6. A grocer bought a certain quantity of sugar for £7. He kept 56 lbs. for family use, and sold the remainder at a profit of ½d. per lb., thereby gaining £1, 3s. 4d. on his outlay. How much sugar did he buy?

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